Classical Gravity from Gluon Interactions

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Zusammenfassung

Diese Arbeit konzentriert sich auf die Doppelkopie-Relation zwischen Eichtheorien und Gravitation sowie ihrer Anwendung in der klassischen Streuung massiver kompakter Objekte. Die Doppelkopie-Relation besagt, dass Observable in einer Gravitationstheorie durch "Quadrieren" entsprechender Größen in einer Eichtheorie abgeleitet werden können. Es ermöglicht die Verwendung moderner Techniken der Eichtheorien, um Probleme wie die Streuung von Schwarzen Löchern in der Gravitation anzugehen.

Wir betrachten zunächst die massive skalare Quantenchromodynamik (SQCD) und führen die Doppelkopie für deren Streuamplituden durch. Aus den resultierenden Amplituden rekonstruieren wir die effektive Lagrange-Funktion. Diese besteht aus einer Graviationstheorie gekoppelt an massive Skalare, ein Axion und ein Dilaton. Zusätzlich erzeugt es auch skalare Selbstwechselwirkungsterme. Der entstehende Lagrangian wird explizit bis zur sechsten Ordnung von Skalarfeldern konstruiert, und es wird eine Form aller Ordnungen postuliert.

Es folgt die Erforschung der Doppelkopie massiver Punktteilchen, die als klassische Version der SQCD-Doppelkopie angesehen werden kann. Die Quellen werden durch Weltlinien-Quantenfeldtheorien formuliert, die mit Yang-Mills, biadjungiertem Skalar und Zwei-Form-Dilaton-Gravitation gekoppelt sind. Wir schlagen eine Doppelkopievorschrift für die eikonalen Phase vor, die verwendet werden kann, um Observablen wie die Impulsablenkung im Streuprozeß abzuleiten und explizit bis zur nächstführenden Ordnung (NLO) zu überprüfen. Wir demonstrieren ferner ihre Beziehung zum klassischen Limes der Streuamplituden und erklären ihre Erweiterung auf die Bremsstrahlung.

Wir untersuchen ferner die nicht-perturbative Doppelkopie klassischer Lösungen. Insbesondere erweitern wir die Kerr-Schild-Abbildung, die es ermöglicht, Lösungen der Einstein-Gleichung aus der Eichtheorie zu erhalten, auf den Fall eines Probeteilchens, das sich im Kerr-Schild-Hintergrund bewegt. Die Umlaufbahnen einer Testladung im nicht-Abelschen Coulomb-Hintergrund und auf der Äquatorialebene des rotierenden Kerr-ähnlichen Hintergrunds werden analysiert und kategorisiert. Wir finden darüberhinaus eine neue Doppelkopie zwischen den erhaltenen Ladungen auf der Eichtheorie und den Gravitationsseiten, die natürlich sowohl für gebundene als auch für ungebundene Zustände funktioniert.

Schließich untersuchen wir die Post-Minkowski'sche (PM) und Post-Newton'sche (PN) Entwicklungen des gravitativen effektiven Drei-Körper-Potentials. Wir liefern auf 2PM Ebene ein formelles nicht-lokales Ergebnis und entwickeln es in der Geschwindigkeit. Wir stellen die Wechselwirkungsterme bis zur Ordnung G^2v^2 wieder her und präsentieren die neuartigen G^2v^4-Beiträg auf 3PN Ebene. Um 2PM-Beiträge zu höherer Ordnung in PN zu erhalten, berechnen wir eine Familie von 3-Punkt-Integralen aus einem Yangian-Bootstrap-Ansatz.

Abstract

This thesis focuses on the double copy relation between gauge theories and gravity and its application in the classical scattering of massive compact objects. The double copy relation states that observables in a gravitational theory can be derived from "squaring" corresponding quantities in a gauge theory. It allows using modern techniques of gauge theories to tackle problems such as black hole scattering in gravity.

We first consider massive scalar quantum chromodynamics (SQCD) and perform the double copy procedure for the scattering amplitudes. We reconstruct the effective Lagrangian from the resulting amplitudes. It yields a gravitational theory of massive scalars coupled to gravity, axion, and dilaton. Additionally, it also produces scalar self-interaction terms. The emerging Lagrangian is constructed explicitly up to the sixth order of scalar fields, and an all-order form is conjectured.

It is followed by exploring the double copy of massive point particles, which can be seen as the classical version of the SQCD double copy. The source objects are formulated by worldline quantum field theories coupled to Yang-Mills, bi-adjoint scalar, and two-form-dilaton-gravity. We propose a double copy prescription for the eikonal phases, which can be used to derive observables such as momentum deflection and check it explicitly up to next-to-leading order (NLO). We demonstrate its relation to the classical limit of scattering amplitudes and explain its extension to classical radiation.

We also investigate the non-perturbative double copy of classical solutions. Specifically, we extend the Kerr-Schild mapping, which allows obtaining solutions of the Einstein equation from that of gauge theory, to the case of a probe particle moving in the Kerr-Schild background. The orbits of a test charge in non-Abelian Coulomb background and on the equatorial plane of the spinning Kerr-like background are analyzed and categorized. We also find a new double copy between the conserved charges on the gauge theory and the gravity sides, which works naturally for both bound and unbound states.

Additionally, we study the Post-Minkowskian (PM) and Post-Newtonian (PN) expansions of the gravitational three-body effective potential. We provide a formal non-local result at 2PM and expand it in the slow-motion limit. We recover the interaction terms up to G^2v^2 and present the novel G^2v^4-contributions at 3PN. To obtain 2PM contributions to higher order in PN, we compute a family of 3-point integrals from a Yangian bootstrap approach.

Publications

This thesis is mainly based on the following peer-reviewed publications [1–4] of the author in various collaborations,

1. J. Plefka, **C. Shi** and T. Wang, *Double copy of massive scalar QCD*, *Phys. Rev. D* **101**, 066004 (2020), arxiv:1911.06785.

2. **C. Shi** and J. Plefka, *Classical Double Copy of Worldline Quantum Field Theory*, *Phys. Rev. D* **105**, 026007 (2022), arxiv:2109.10345.

3. R. Gonzo and **C. Shi**, *Geodesics From Classical Double Copy*, *Phys. Rev. D* **104**, 105012 (2021), arxiv:2109.01072.

4. F. Loebbert, J. Plefka, **C. Shi** and T. Wang, *Three-body effective potential in general relativity at second post-Minkowskian order and resulting post-Newtonian contributions*, *Phys. Rev. D* **103**, 064010 (2021), arxiv:2012.14224.

The following publications [5] of the author during his doctoral is not discussed in detail in this thesis. However, a comment is made on [5] in section 3.5.

5. J. Plefka, **C. Shi**, J. Steinhoff and T. Wang, *Breakdown of the classical double copy for the effective action of dilaton-gravity at NNLO*, *Phys. Rev. D* **100**, 086006 (2019), arxiv:1906.05875.

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Contents

Chapter 1

Introduction

In our modern understanding of the universe, gauge theory and general relativity play central roles in formulating physical phenomena. The strong, weak, and electromagnetic forces are well described by Yang-Mills (YM) gauge theories. They are quantum consistent at the microscopic level and have made high precision predictions of particle scattering in the Standard Model, which successfully meet measurements of collision experiments. For instance, the spontaneous symmetry breaking of the electroweak $SU(2) \otimes U(1)$ theory predicts the existence of a Higgs boson, which was found in the Large Hadron Collider at CERN [6, 7]. Gravity, on the other hand, is distinct in the sense that it dominates the long-distance interaction of macroscopic objects and is modeled in a geometric way by Einstein's general relativity (GR). It explains an extensive range of phenomena, such as the formation of the solar system, the existence of black holes and gravitational waves, and describes the dynamics of the cosmos. With modern advanced telescopes, the Event Horizon Telescope Collaboration can directly take pictures of black holes and their vicinity [8]. However, it is well known that general relativity is incompatible with quantum mechanics in the high-energy (short distance) limit. Seeking a formulation of quantum gravity remains an open question. It is one of the most crucial issues in physics.

Despite the apparent distinctions between gauge and gravity theories, there is a surprising connection between the two classes of theories, known as the *double copy* relation, which allows making predictions in gravity directly from gauge theories. Although in this case, one usually considers some extensions of gravity, usually with some supersymmetries. The double copy could lead to a novel conceptual understanding of the unification of gauge theory and gravity and to new technical methods that drastically simplify computations in gravitational observables.

The gauge/gravity double copy relation simply states that in some sense, gravity theories can be seen as the squaring of gauge theories, often sketched as[1],

$$(\text{Gravity}) \sim (\text{gauge theory}) \otimes (\text{gauge theory}).$$

This is initially discovered for scattering amplitudes in string theory. The tree-level closed string amplitudes are found to be expressible as a sum over products of gauge-invariant open string tree amplitudes and a so-called "kernel", known as the KLT relation after Kawai, Lewellen, and Tye [9]. In the low energy limit, the open string is reduced to Yang-Mills

[1] Note that by gravity, we usually refer to some generalizations of Einstein's gravity with additional massless fields such as dilaton, B-field, and sometimes with additional symmetries.

theories, and the closed string gives Einstein gravity coupled to a dilaton and an anti-symmetric B-field, usually referred to as the $\mathcal{N}=0$ supergravity. Hence, the KLT relation is transferred to the amplitudes of gauge and $\mathcal{N}=0$ gravity. In 2013, Bern, Carrasco, and Johansson (BCJ) discovered that the double copy was a result of the duality between color factors and the kinematic numerators [10, 11], which says that it is possible to rearrange the gauge-dependent kinematic numerators to satisfy the same algebraic equations with the color factors in Yang-Mills theory. One can then simply replace the color factors with the numerators to obtain the amplitudes in gravity. At tree level, the color-kinematics duality is strictly proven [12–17], and the double copy holds as a consequence. Although it is not entirely proven for loop amplitudes, many explicit pieces of evidence exist at the level of integrands [11,18–38]. For recent reviews on the double copy relation, see, for example [39,40]

Scattering amplitudes have always been at the heart of making predictions from quantum field theories. However, the traditional approach calculation requires drawing all Feynman diagrams, the number of which proliferates in the number of external legs and loop order. Despite the complexity, the final results could be much simpler than the intermediate steps. For example, the tree-level maximal helicity violating (MHV) all-gluon amplitude fits into one line given by the Parke-Taylor formula [41]. Exploiting the fact that amplitudes are gauge-independent and on-shell, modern techniques, such as generalized unitarity, on-shell recursion, and spinor-helicity formalism, allow to land on the final answers quickly. The gravitational scatterings are even more complicated compared to YM ones due to the higher-ranked tensor structures. The double copy thus provides a fast way to amplitudes in gravity theories, which can be further related to the classical observable of compact astronomical objects.

Since the LIGO/Virgo detectors observed gravitational waves for the first time in 2015 [42], astronomy and cosmology research has entered a new era of multi-messengers. The next-generation detectors require having high precision predictions of gravitational waveforms, especially for future observatories. The past few years have witnessed the success of using our knowledge of scattering amplitudes for high-energy particle collisions to higher-order calculations in the perturbative regime of the two-body problem. For recent reviews, see [40,43]. The connection of quantum amplitudes to classical observables is systematically formulated in an on-shell method developed by Kosower, Maybee, and O'Connell (KMOC) [44]. They carefully analyze the length scale in scattering events in the classical limit. Take scattering binaries as an example. The masses are required to be much larger than the Planck mass $m_1, m_2 \gg M_{\text{Planck}}$, and the impact parameter should be much larger than the Schwarzschild radius, which should be larger than the de Broglie wavelength, $|\mathbf{b}| \gg 2Gm_i/c^2 \gg \lambda$. In the momentum space, the latter is equivalent to requiring the momentum transfer to scale as the Planck constant $|\mathbf{q}| \sim \hbar|\mathbf{q}|$, thus much smaller than the incoming/outgoing momenta $|\mathbf{p}|$ similar to the Regge limit. The classical observables are taken as this classical limit of the expectation values of the quantum operator evaluated with coherent states, which are then associated with corresponding amplitudes. The conservative dynamics of binaries are related to four-point amplitudes of two pairs of massive external legs, and the radiation can be computed from five-point amplitudes with an additional massless outgoing line.

Recent progress in calculating the dynamics of inspiral binaries in the post-Minkowskian (PM) regime using amplitude techniques has taken advantage of on-shell methods, the double copy, and effective field theories. For example, the tree-level gravitational amplitudes are computed at the third PM order via the double copy of corresponding YM amplitudes [45]. Then the classical limit is taken to discard quantum contributions, drastically simplifying the expressions. The tree amplitudes are glued together by the generalized unitarity method, giving the classical loop-level contributions. After performing the integration using techniques such as the separation of regions and integration by parts, the results are then matched to effective field theory Lagrangian, yielding the off-shell potential, which is expected to be applied to also bound states.

It is worth mentioning that from a conceptual perspective, the double copy also provides fascinating insights into the gauge invariant and on-shell theory, as well as implicit connections between superficially distinct theories. The double copy relation goes beyond Yang-Mills and $\mathcal{N} = 0$ gravity. The simplest example is the bi-adjoint scalar theory (ϕ^3 theory), whose amplitudes carry two copies of color factors and can be obtained by replacing the kinematic numerators in the YM amplitudes with the color factors. Hence, it is usually seen as the *zeroth copy* of YM theory. Furthermore, since the double copy of bi-adjoint and YM gives YM theory again, it is often treated as an identity in the double copy construction. Many other theories can be related by the double copy relations. Suppose one turns on supersymmetries in the gauge theory side and notices that the two copies of kinematic numerators do not necessarily come from the same theory. In that case, one can construct many gravity theories with supersymmetries. For instance, the product of the $\mathcal{N} = 0$ super YM (pure YM) coupled to n_s adjoint scalars and $\mathcal{N} = 4$ super YM ($\mathcal{N} = 4$ SYM) yields the $\mathcal{N} = 4$ supergravity, and the square of $\mathcal{N} = 4$ SYM gives $\mathcal{N} = 8$ supergravity. The double copy relations are even applicable to various effective field theories. Some famous cases are: the double copy of two non-linear sigma models (NLSM) is the special Galileon theory; the product of the NLSM and the ϕ^3 theory coupled to YM gives the Dirac-Born-Infeld coupled to NLSM; the product of YM and YM+ϕ^3 results in YM theory coupled to $N = 0$ supergravity [46, 47]. The double copy is now understood as a property of many classes of theories. They form a large web related by sharing common gauge-theory factors [39, 40]. Another formulation of the double copy is given by Cachazo, He, and Yuan (CHY) [48–50]. In this formalism, tree-level massless amplitudes are computed similarly to string theory - an n-point amplitude is written as an integral over n punctures on the Riemann sphere, supported by the solutions to the scattering equations. The double copy is realized simply by taking the product of two integrands of the single copy theories, making it the integrand of the generated double copy theory.

Many efforts are also trying to perform the double copy procedure directly at the classical level. Most notable is the double copy of exact solutions to the equations of motion, first noticed by Monteiro, O'Connell, and White [51]. The Kerr-Schild type solutions to the Einstein equation can be related to the *single copy* solutions of the Maxwell theory. The simplest example is the correspondence of the Schwarzschild metric and the Coulomb potential. Less trivially, the Kerr solution is mapped to a particular $\sqrt{\text{Kerr}}$ potential in Yang-Mills. The classical double copy is then further generalized to solutions of Petrov type D and non-twisting type N, where the Weyl spinors of the solutions can be factorized into products of Maxwell spinors of solutions of gauge theory, and it is strictly proven using twistorial techniques [52–56]. Many works follow to develop the classical double copy [57–75].

In this thesis, we focus on the double copy relation between YM and gravity theory, stressing its application to classical gravitation physics, but we are also interested in getting effective potential directly in the classical regime. It consists of four main chapters. Chapter 2 investigates the extension of the YM double copy to include massive spinless quantum scalar fields, the quantum version of the massive compact objects interacting via YM/gravity. We follow the traditional approach by computing all possible amplitudes up to quartic order in the coupling constants (next-to-leading order). We check the color-kinematics duality explicitly and use double copy to obtain the amplitudes for the gravitational theory. We match the amplitudes to the proposed Lagrangian up to six-scalar interactions, which couples two-form-dilaton-gravity to massive scalar fields. We found that the resulting Lagrangian contains contact terms of the massive scalars, which are short-range interactions and confirms that they have no contributions to classical physics.

In chapter 3, we take advantage of the recently developed worldline quantum field theory (WQFT) formalism to examine the double copy at the classical level akin to amplitudes. WQFT allows for computing classical observables in an efficient diagrammatic way. We consider the WQFTs of massive (charged) point particles described by a worldline action coupled to a bi-adjoint scalar, Yang-Mills field, and dilaton-gravity. We establish a classical double copy relation in these WQFTs for the eikonal phase and the classical observables (deflection, radiation). The bi-adjoint scalar field theory fixes the locality structure of the double copy from Yang-Mills to dilaton-gravity. The eikonal scattering phase (or free energy of the WQFT) is computed to next-to-leading order (NLO) in coupling constants using the double copy and directly finding complete agreement. We clarify the relation of our approach to previous studies in the effective field theory formalism. Finally, the equivalence of the WQFT double copy to the double copy relation of the classical limit of quantum scattering amplitudes is shown explicitly up to NLO.

In chapter 4, we seek to extend the Kerr-Schild double copy to the case where a test particle moves in static backgrounds. Establishing a mapping relation between the conserved quantities in the YM and gravity potentials, we consider two exceptional cases: the non-spinning Schwarzschild and the equatorial plane of the spinning Kerr background. We examine the classes of orbits in the YM single copy of these two situations. The trajectories feature circular, elliptic, plunge, and hyperbolic behaviors, just as the corresponding geodesic motions at the gravity side. The mapping of the conserved charges allows for recovering the full geodesic equations from the YM ones. Our double copy mapping naturally works for both bound and hyperbolic states. By contrast, the scattering double copy works only for the latter.

Chapter 5 applies worldline effective field theory to the classical three-body problem. We calculate the integrand of the effective potential up to second order in the post-Minkowskian expansion. We give formal results by carefully analyzing the separations of the coordinates of the three bodies and performing integration in each case. In order to better compute the post-Newtonian (PN) contributions, we perform slow velocity expansion at the integrand level. The integrals are bootstrapped by exploiting the Yangian symmetry in the dimensional regularization scheme in 3D. We successfully reproduce the Dirac-Infeld-Hoffmann interaction term and find agreement with literature at 2PN in the two-body limit. The new G^2v^4-contributions at 3PN are explicitly presented, and we outline the generalization to G^2v^{2n}.

1.1 The Double Copy Relation

In this section, we briefly introduce the double copy relation, focusing on the Bern-Carrasco-Johansson (BCJ) double copy of the amplitudes of the gauge and gravity [10, 11]. We will first consider the pure YM theory with the Lagrangian

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a}, \tag{1.1}$$

where the field strength reads

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - igf^{abc}A^b_\mu A^c_\nu, \tag{1.2}$$

with g being the coupling constant. The generators of the gauge group are normalized such that

$$[T^a, T^b]_{ij} = f^{abc}T^c_{ij}, \qquad \mathrm{Tr}(T^aT^b) = \frac{\delta^{ab}}{2}. \tag{1.3}$$

Note that our structure constant f^{abc} is different from the more common convention by a factor of $-i$. An n-point L-loop amplitude in D dimensions can be formally written as

$$A^{(L)}_n = (ig)^{2L+n-2}\int\prod_{j=1}^{L}\frac{\mathrm{d}^D p_j}{(2\pi)^D}\sum_{i\in\mathrm{Trivalent}}\frac{c_i n_i}{S_i D_i}, \tag{1.4}$$

where c_i are the color factors that are products of structure constants and group generators, n_i are the kinematic numerators which are dependent on the external momenta and polarization vectors, $D_i = \prod_i d_i$ are the denominators which are a product of propagators $d_i \sim p_i^2$. S_i are the symmetry factors associated with the diagram, which are trivial for tree amplitudes but essential to removing overcount of diagrams. It is important to note that the sum is over all the three-point diagrams. We tear apart diagrams with contact interactions by multiplying and dividing by certain propagators and we attribute them to trivalent diagrams. Note that there is no unique way of doing this separation.

The decomposition (1.4) is over-completed because the generators and constant structures satisfy the Jacobi identity,

$$f^{abe}f^{cde} + f^{bce}f^{ade} + f^{cae}f^{bde} = 0 \tag{1.5}$$

$$T^a_{ik}T^b_{kj} - T^b_{ik}T^a_{kj} = f^{abc}T^c_{ij}. \tag{1.6}$$

This leads to the consequence that some of the color factors are related by algebraic equations

$$c_i + c_j + c_k = 0 \tag{1.7}$$

Due to the over-completeness of the basis, the definition of the kinematic numerators n_i in (1.4) is not unique. The key point of the color-kinematics is that it is possible to rearrange the numerators n_i such that they satisfy exactly the same equations with the corresponding color factors c_i [10, 11],

$$n_i + n_j + n_k = 0. \tag{1.8}$$

This can be achieved by performing the generalized gauge transformations[2]

$$n_i \to n_i + \Delta_i, \tag{1.9}$$

with the variation Δ_i leaving the amplitude invariant

$$\sum_{i\in\mathrm{Trivalent}}\frac{c_i\Delta_i}{S_i D_i} = 0. \tag{1.10}$$

[2]A usual gauge transformation shift the numerators by replacing the polarization vectors with the momenta, but a generalized gauge transformation shift the numerators in any possible way as long as (1.10) is satisfied.

Numerators that satisfy these relations are sometimes referred to as BCJ numerators. With the color-kinematics duality, we can now simply do the replacement

$$c_i \quad \to \quad n_i \tag{1.11}$$

to obtain the emerging amplitude that also satisfies generalized gauge transformation. Since the power of the polarization is doubled, this is a hint that the resulting amplitude is for a theory with a tensor field of rank-two, which is the same as the spacetime metric. Remarkably, we can take two sets of BCJ numerators n_i and $\tilde{n}_i$ from different gauge theories. The amplitude of the double copy gravity theory of the two gauge theories is obtained as

$$M_n^{(L)} = \left(\frac{-i\kappa}{4}\right)^{2L+n-2} \int \prod_{j=1}^{L} \frac{\mathrm{d}^D p_j}{(2\pi)^D} \sum_{i\in\text{Trivalent}} \frac{n_i \tilde{n}_i}{S_i D_i}, \tag{1.12}$$

where $\kappa^2 = 32\pi G$ is the gravitational coupling constant. Since the numerators satisfy the same algebraic relations as the color factors, we can perform the generalized gauge transformation of one set of the numerators,

$$\tilde{n}_i \;\to\; \tilde{n}'_i = \tilde{n}_i + \Delta_i, \qquad \text{with} \qquad \sum_{i\in\text{Trivalent}} \frac{n_i \Delta_i}{S_i D_i} = 0. \tag{1.13}$$

The new $\tilde{n}'_i$ generically will not respect BCJ duality, but the amplitudes will be invariant. This implies that in practice we do not need to arrange both sets of numerators to satisfy color-kinematics duality, but only one set is enough. The other could be related to a set of BCJ numerators by a generalized gauge transformation, just like the above $\tilde{n}'_i$. By doing so, we will drastically simplify the construction. However, we should keep in mind that both gauge amplitudes should, in principle, respect color-kinematics duality.

The color-kinematics duality at tree level for YM is strictly proven. However, less is known at loop level, so (1.8) remains a conjecture. It is important to note that for the double copy at loop levels to work, we have to keep the color factor generic, i.e., not explicitly use properties such as antisymmetry or other identities specific to the gauge group. Otherwise, some of the color factors will vanish, and we will lose track of the corresponding numerators. We will run into a similar problem in 3, where we will avoid the problem by introducing extra flavors of matters.

We have already seen that the construction gives amplitudes respecting generalized gauge transformation. Even more crucial is that the emerging amplitude is invariant under linearized diffeomorphisms, which is a consequence of the gauge-invariance of the two gauge amplitudes. In order to see this, let us consider a gauge transformation concerning only one external gluon,

$$\epsilon_\mu(p) \;\to\; \epsilon_\mu(p) + p_\mu. \tag{1.14}$$

It will be helpful to extract the polarization vector from the numerators

$$n_i = \epsilon_\mu n_i^\mu, \qquad \tilde{n}_i = \epsilon_\mu \tilde{n}_i^\mu. \tag{1.15}$$

Since the amplitude is gauge invariant, we have

$$\sum_{i\in\text{Trivalent}} \frac{c_i (p_\mu n_i^\mu)}{S_i D_i} = 0. \tag{1.16}$$

Suppose in the construction (1.12), both sets of numerators satisfy the same equation as the color factors c_i. Since the above equation depends only on the generic relations of c_i, the following has to be correct,

$$\sum_{i\in\text{Trivalent}} \frac{n_i(p_\mu \tilde{n}_i^\mu)}{S_i D_i} = \sum_{i\in\text{Trivalent}} \frac{\tilde{n}_i(p_\mu n_i^\mu)}{S_i D_i} = 0. \tag{1.17}$$

Taking the traceless-transverse gauge, we identify

$$\epsilon_{\mu\nu} := \epsilon_{((\mu}\epsilon_{\nu))} \tag{1.18}$$

as the polarization tensor of the graviton, where the double brackets denote the symmetric-traceless part. The linearized diffeomorphism acting on one external leg results in

$$\epsilon_{\mu\nu} \;\to\; \epsilon_{\mu\nu} + p_{(\mu}q_{\nu)} \tag{1.19}$$

where q_μ is an arbitrary auxiliary null vector that satisfies $p\cdot q = 0$, and the brackets of the Lorentz indices denote the symmetric part. Under this transformation, the double copy amplitude becomes

$$M_n^{(L)} \;\to\; M_n^{(L)} + \int \frac{(\mathrm{d}^D p)^L}{(2\pi)^{LD}} \sum_{i\in\text{Trivalent}} \left[\frac{(p_\mu n_i^\mu)\tilde{n}_i\big|_{\tilde{\epsilon}_\nu\to q_\nu}}{S_i D_i} + \frac{(p_\mu \tilde{n}_i^\mu) n_i\big|_{\epsilon_\nu\to q_\nu}}{S_i D_i} \right]. \tag{1.20}$$

Since the Jacobi identities of the numerators do not depend on properties of the polarization vectors, they must still be valid under the replacement $\epsilon_\nu \to q_\nu$,

$$(n_i + n_j + n_k)\big|_{\epsilon_\nu\to q_\nu} = 0, \qquad (\tilde{n}_i + \tilde{n}_j + \tilde{n}_k)\big|_{\tilde{\epsilon}_\nu\to q_\nu} = 0. \tag{1.21}$$

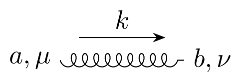
Therefore the last two terms in (1.20) are vanishing, confirming that the double copy amplitudes are indeed invariant under linearized diffeomorphism and the emerging theory is a gravity theory.

1.2 Four-point Example

Let us now demonstrate the details of the double copy procedure by taking the four-point amplitude as an example. We will follow the textbook approach to compute the amplitudes: we draw all Feynman diagrams and sum over them. We supplement the pure YM Lagrangian (1.1) with the Feynman gauge condition

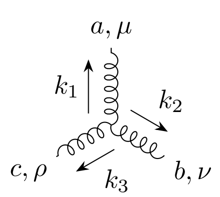
$$\mathcal{L}_{\text{gh}} = -\frac{1}{2}(\partial^\mu A_\mu^a)^2. \tag{1.22}$$

This gauge gives canonically normalized propagators



$$= \frac{-i}{k^2 + i\varepsilon}\eta_{\mu\nu}\delta^{ab}. \tag{1.23}$$

The three- and four-gluon vertices are



$$= igf^{abc}V_{123}^{\mu\nu\rho}, \tag{1.24}$$

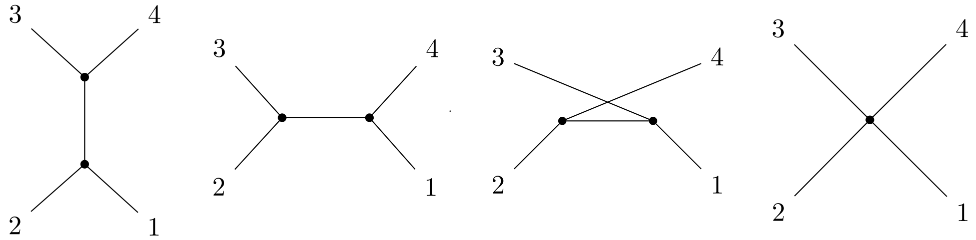
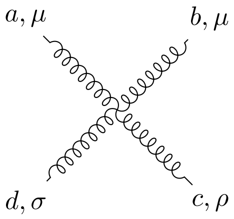


Figure 1.1: Feynman diagrams contributing to the four-point amplitudes.



$$
\begin{aligned}
&= ig^2[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\
&\quad + f^{bce}f^{ade}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}) \\
&\quad + f^{cae}f^{bde}(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma})].
\end{aligned}
\tag{1.25}
$$

where we have defined a shorthand for the kinematic part as

$$
V_{123}^{\mu\nu\rho} = \big[\eta^{\mu\nu}(k_1-k_2)^\rho + \eta^{\nu\rho}(k_2-k_3)^\mu + \eta^{\rho\mu}(k_3-k_1)^\nu\big]. \tag{1.26}
$$

The four-gluon amplitude consists of four diagrams, see figure 1.1. As stated in (1.4), we express it as a sum over three channels

$$
A_4^{(0)} = -g^2\Big(\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}\Big), \tag{1.27}
$$

where $s = (p_1 + p_2)^2, t = (p_1 + p_3)^2, u = (p_1 + p_4)^2$ are the Mandelstam variables. The color factors are derived from the Feynman diagrams,

$$
c_s = f^{abe}f^{cde}, \qquad c_t = f^{bce}f^{ade}, \qquad c_u = f^{cae}f^{bde}. \tag{1.28}
$$

They are not linearly independent, but satisfy the following Jacobi identity

$$
c_s + c_t + c_u = 0. \tag{1.29}
$$

The numerator in the s-channel reads

$$
\begin{aligned}
n_s = &- i\,(\epsilon_1\cdot\epsilon_2(p_1 - p_2)^\rho + 2\epsilon_1\cdot k_2\,\epsilon_2^\rho - 2\epsilon_2\cdot k_1\,\epsilon_1^\rho)\,(\epsilon_3\cdot\epsilon_4(p_3 - p_4)_\rho + 2\epsilon_3\cdot k_4\,\epsilon_{4,\rho} - 2\epsilon_4\cdot k_3\,\epsilon_{3,\rho}) \\
&- is\,(\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4 - \epsilon_1\cdot\epsilon_4\,\epsilon_2\cdot\epsilon_3)\,,
\end{aligned}
\tag{1.30}
$$

where the first line is from the first diagram with YM Feynman rules in figure 1.1. The second line is due to the fact that we need to attribute the contact interaction, the last diagram in figure 1.1, to the three channels according to the color factors. To add it to the numerator, we also need to multiply and divide it by the corresponding Mandelstam variable, which is s in this case. The other two channels can be simply obtained by relabeling $(1, 2, 3)$,

$$
n_t = n_s\big|_{1\to2,\,2\to3,\,3\to1,\,s\to t} \qquad n_u = n_u\big|_{1\to3,\,2\to1,\,3\to2,\,s\to u}. \tag{1.31}
$$

It is straightforward to check that the three numerators satisfy a identity akin to (1.29)

$$
n_s + n_t + n_u = 0. \tag{1.32}
$$

With this color-kinematics duality, we are now ready to obtain the double copy amplitude according to (1.12),

$$M_4^{(0)} = -\frac{\kappa^2}{16}\left(\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}\right). \tag{1.33}$$

We can now explicitly check the linearized diffeomorphism,

$$M_4^{(0)}\Big|_{\epsilon_\mu\epsilon_\nu \to p_{(\mu}q_{\nu)}} = 0 \tag{1.34}$$

by using on-shell conditions $p^2 = 0$, $\epsilon \cdot p = 0$ and $p \cdot q = 0$. In order to verify that the resulting amplitude is indeed for four-graviton scattering, we need to treat general relativity as a low energy effective field theory and consider the expansion around Minkowskian space time.

1.3 Post-Minkowskian Expansion of Gravity

In this thesis, we are mainly interested in the situation where the interactions are weak so that we can treat the problem perturbatively. In the case of gravity, we thus expand the spacetime around flat background. This is referred to as the post-Minkowskian (PM) limit. It is characterized by Newton's constant G, or equivalently, the gravitational coupling constant $\kappa = \sqrt{32\pi G}$. Special relativistic effects are preserved in the Minkowskian background, so at fixed order in PM expansion, quantities have exact dependence on the velocity. These properties are similar to those of scattering amplitudes, which implies that we should treat gravity as an effective field theory and consider its perturbative expansion.

It is well-known that general relativity is governed by the Einstein-Hilbert action

$$S_{\mathrm{EH}} = -\frac{2}{\kappa^2}\int \mathrm{d}^4x\sqrt{|g|}R, \tag{1.35}$$

where $g = \mathrm{Det}(g_{\mu\nu})$ is the determinant of the metric. As stated before, in perturbation theory, we expand the metric around flat spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}, \tag{1.36}$$

where $\eta_{\mu\nu}$ is the Minkowskian metric. We use the mostly minus signature throughout this thesis. In quantum gravity nomenclature, $h_{\mu\nu}$ is usually referred to as the graviton, which we will follow this convention hereafter. We can derive the expansion of the inverse metric and the determinant,

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda}h^{\nu}{}_{\lambda} + \mathcal{O}\left(\kappa^3\right), \tag{1.37}$$

$$g = \mathrm{Det}(\eta_{\mu\nu}) - \kappa h^{\mu}{}_{\mu} + \frac{1}{2}\kappa^2\left(h_{\mu\nu}h^{\mu\nu} - h^{\mu}{}_{\mu}h^{\nu}{}_{\nu}\right) + \mathcal{O}\left(\kappa^3\right). \tag{1.38}$$

The indices will be raised and lowered by $\eta_{\mu\nu}$. Since general relativity respects gauge diffeomorphism invariance, we should fix the gauge freedom by including a Faddeev-Popov term S_{gf} to the action. There are various choices of gauge in the literature. In this chapter, we will adopt the commonly used de Donder gauge

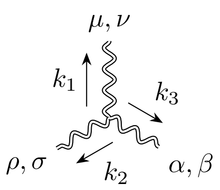
$$S_{\mathrm{gf}} = \int \mathrm{d}^4x\left(\partial_\nu h^{\mu\nu} - \frac{1}{2}\partial^\mu h^{\nu}_{\nu}\right)^2. \tag{1.39}$$

We note that in the case of dilaton gravity in chapter 3, we will use another gauge to our convenience. Since we are interested merely in classical physics, we do not need to be concerned about ghosts.

In de Donder gauge, we can derive the Feynman rule of the graviton propagator

$$\mu,\nu \;\sim\!\sim\!\sim\!\sim\; \rho,\sigma = \frac{i}{k^2+i\varepsilon}\left(\eta^{\mu(\rho}\eta^{\sigma)\nu} - \frac{1}{D-2}\eta^{\mu\nu}\eta^{\rho\sigma}\right), \tag{1.40}$$

where the parenthesis around Lorentz indices denotes symmetrization with unit weight, e.g. $v^{(\mu}w^{\nu)} = \frac{1}{2}(v^\mu w^\nu + v^\nu w^\mu)$ for arbitrary tensors v^μ, w^μ. The weak-field expansion produces self-interaction terms with arbitrarily many gravitons, which are very lengthy and cumbersome. For instance, the cubic vertices of three-graviton is given as



$$= \tag{1.41}$$

$$\begin{aligned} i\kappa\,\mathrm{Sym}\Big[&-\frac{1}{4}k_1\cdot k_2\eta_{\mu\nu}\eta_{\rho\sigma}\eta_{\alpha\beta} + k_{1\rho}k_{1\beta}\eta_{\mu\nu}\eta_{\sigma\alpha} - \frac{1}{2}k_{1\sigma}k_{2\mu}\eta_{\nu\rho}\eta_{\alpha\beta} + \frac{1}{4}k_1\cdot k_2\eta_{\mu\rho}\eta_{\nu\sigma}\eta_{\alpha\beta} \\ &+ \frac{1}{2}k_{1\alpha}k_{2\beta}\eta_{\mu\rho}\eta_{\nu\sigma} + k_1\cdot k_2\eta_{\mu\nu}\eta_{\rho\alpha}\eta_{\sigma\beta} + k_{1\alpha}k_{1\beta}\eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{2}k_{1\rho}k_{1\sigma}\eta_{\mu\nu}\eta_{\alpha\beta} \\ &+ 2k_{1\rho}k_{2\beta}\eta_{\sigma\mu}\eta_{\nu\alpha} - k_1\cdot k_2\eta_{\nu\rho}\eta_{\sigma\alpha}\eta_{\beta\mu} + k_{1\rho}k_{2\mu}\eta_{\sigma\alpha}\eta_{\beta\nu} + \text{permutations}\,(1,2,3)\Big], \end{aligned}$$

where "Sym" denotes the symmetrization of Lorentz indices associated with the same graviton, and one needs to consider all possible permutations of all the terms. Counting all the permutations, there are more than 60 terms in the "simplest" graviton vertex. Let alone to mention the four-graviton vertex which will be needed in the four-point amplitude. Many modern techniques, such as on-shell BCFW recursion relation [76], can circumvent the difficulties. For recent reviews on scattering amplitude calculations, please see [40, 77, 78]. However, this topic is beyond the scope of this thesis. Fortunately, with the power of modern computer hardware and software, it is possible to attack the difficult problem with brute force. For instance, using *Mathematica* with the *xAct* package, we can get the four-graviton amplitude in seconds by adding all the Feynman diagram given in figure 1.1 with gravitational Feynman rules. It exactly matches the double copy four-point amplitude (1.33). In this simple example, we can see the power of the double copy - it simplifies the calculation significantly compared to traditional approaches.

1.4 Post-Newtonian Expansion of Gravity

In the case of massive objects interacting via gravity, we can consider another limit that breaks special relativity. In this approximation, we recover Newton's law of gravitation in the leading order, hence it is called post-Newtonian expansion. It could be understood as a refinement of the PM expansion - on top of the weak field limit, we also consider the velocity of the motion to be small compared to the speed of light. In terms of the relativistic velocity u_i of particle i, it means

$$u_i^\mu = \left(1, \frac{\mathbf{v}_i}{c}\right), \tag{1.42}$$

	0PN	1PN	2PN	3PN	4PN	...
1PM	1	$\mathbf{v}^2$	$\mathbf{v}^4$	$\mathbf{v}^6$	$\mathbf{v}^8$	...
2PM		1	$\mathbf{v}^2$	$\mathbf{v}^4$	$\mathbf{v}^6$	...
3PM			1	$\mathbf{v}^2$	$\mathbf{v}^4$	...
4PM				1	$\mathbf{v}^2$	...
			$\vdots$			

Figure 1.2: The relation between PM and PN expansion.

with $\mathbf{v}_i$ being the spacial component of the velocity. We have reintroduced the inverse of the speed of light c^{-1} as the counting parameter of PN expansion. Therefore, nPN corresponds to order $\mathcal{O}(c^{-2(1+n)})$.

This expansion originates from bound binaries, where the third Kepler law, or the virial theorem, then tells us that

$$\frac{\mathbf{v}_r^2}{c^2} \sim \frac{1}{c^2}\frac{\kappa^2(m_1+m_2)}{32\pi r}, \tag{1.43}$$

where r is the distance between the two objects with mass m_1, m_2, and $\mathbf{v}_r$ is the relative velocity. In (1.43), we can see the interplay of the PM and PN approximation - the weak-field coupling constant scales as

$$\kappa \sim \frac{\kappa}{c}. \tag{1.44}$$

Therefore, PN expansion is in fact a double expansion in both the gravitational constant and velocity of the particles, as illustrated in figure 1.2. The PN approximation also results in a non-relativistic expansion of the massless scalar propagator in the so-called potential region $\omega := k^0 \ll |\mathbf{k}|$, reads

$$\frac{1}{k^2} = \frac{1}{\omega^2 - \mathbf{k}^2} = -\sum_{\alpha=1}^{\infty} \frac{\omega^{2\alpha-2}}{(\mathbf{k}^2)^\alpha}\,. \tag{1.45}$$

Note that we have dropped the $i\varepsilon$ since it does not matter in the conservative sector, which will be of our main interest in the PN limit. In coordinate space, the propagator reads

$$D(x_{ij}) := \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}e^{ik\cdot x_{ij}} = -\int \frac{\mathrm{d}^Dk}{(2\pi)^D}e^{i\mathbf{k}\cdot\mathbf{r}_{ij}}\sum_{\alpha=0}^{\infty}\frac{(-1)^\alpha \partial_{t_i}^{2\alpha}\delta(t_{ji})}{c^{2\alpha}(\mathbf{k}^2)^{\alpha+1}}\,, \tag{1.46}$$

where $x_{ij}^\mu := x_i^\mu - x_j^\mu$ is the difference between two spacetime events, and $t_{ij} := t_i - t_j, \mathbf{r}_{ij} := x_i - x_j$ represents the time and spatial components, respectively. We have performed the energy (ω) integral in last equal sign. Hence, with the expression for the D-dimensional Fourier transform of the momentum space propagator

$$\int \frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(\mathbf{k}^2)^\alpha} = \frac{1}{4^\alpha \pi^{D/2}}\frac{\Gamma_{D/2-\alpha}}{\Gamma_\alpha}r^{2\alpha-D}, \tag{1.47}$$

and doing integration by parts, in coordinate space, we have

$$D(x_{ij}) = -\frac{1}{4\pi}\left(\frac{\delta(t_i-t_j)}{r_{ij}} - \frac{r_{ij}}{2c^2}\partial_{t_i}\partial_{t_j}\delta(t_i-t_j) + \frac{r_{ij}^3}{24c^4}\partial_{t_i}^2\partial_{t_j}^2\delta(t_i-t_j)\right) + \mathcal{O}(c^{-4}), \tag{1.48}$$

with $r_{ij} = |\mathbf{r}_{ij}|$.

1.5 Classical Double Copy

As mentioned at the beginning of this chapter, some exact solutions in YM and general relativity are related to each other in a double copy construction. The idea was pioneered by Monteiro, O'Connell, and White [51], who discovered that classical solutions of *Kerr-Schild* type in Einstein gravity could be related to some "single copy" solutions of pure gauge theory. Classical solutions of Kerr-Schild type of Einstein equation in the vacuum admit the following form

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu} - 2\,GM\varphi(x)k_\mu(x)k_\nu(x), \tag{1.49}$$

where G is Newton's constant, k_μ is null with respect to both the Minkowski metric $\bar{g}_{\mu\nu}$ and the curved metric $g_{\mu\nu}$, and $\varphi(x)$ is a scalar function.[3] This type of metric has the advantage that it linearizes the Ricci tensor. With this decomposition, it can be proven that a gauge field of the form

$$A^a_\mu(x) = \frac{g}{4\pi}\tilde{c}^a\varphi(x)k_\mu(x) \tag{1.50}$$

is a solution to the equations of motion of YM theory in the vacuum. In (1.50) we use $\tilde{c}^a$ to denote a static charge that acts as a source of the field. However, in this case, if we replace it with an electric charge, we get a solution to Maxwell's theory.

The simplest example is the Schwarzschild metric in the Eddington-Finkelstein coordinates $x^\mu(\tau) = (t, r, \theta, \phi)$,

$$\mathrm{d}s^2 = \left(\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\right) - \frac{2GM}{r}\left(\mathrm{d}t + \mathrm{d}r\right)^2 \tag{1.51}$$

with $\mathrm{d}\Omega^2$ being the 2-dimensional sphere metric. Comparing it to (1.50), we can identify

$$\varphi(x) = \frac{1}{r}, \qquad k_\mu = (1, 1, 0, 0). \tag{1.52}$$

It is straightforward to obtain the single copy Yang-Mills field

$$A^a_t = A^a_r = \frac{g}{4\pi}\frac{\tilde{c}^a}{r}, \qquad A^a_\phi = A^a_\theta = 0, \tag{1.53}$$

which in electromagnetism is nothing but the Coulomb potential. We will refer to this gauge background as $\sqrt{\text{Schw}}$ in the following since, as we will see, it corresponds to the Schwarzschild metric in the classical double copy relation.

Another notable solution in general relativity is the Kerr metric with a spinning compact object as a source. In Kerr-Schild coordinates, we have

$$\varphi(x) = \frac{r^3}{r^4 + a^2z^2}, \qquad k_\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right). \tag{1.54}$$

where r is defined implicitly through the following constraints

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \qquad \forall (x, y, z) \in \mathbb{R}^3\backslash\{x^2 + y^2 \leq a^2,\ z = 0\} \tag{1.55}$$

$$r = 0 \qquad \forall (x, y, z) \in \{x^2 + y^2 \leq a^2,\ z = 0\}, \tag{1.56}$$

[3] Note that in this chapter, we will use $\bar{g}_{\mu\nu}$ to denote the Minkowski metric in general coordinates.

and the magnitude of a is the spin length of the black hole, (x, y, z) is a set of Cartesian coordinate. The field is singular on a ring of radius a in the $x - y$ plane. We can directly write down the single copy $\sqrt{\text{Kerr}}$ field

$$A^a_\mu = \frac{g}{4\pi}\frac{r^3\tilde{c}^a}{r^4 + a^2 z^2}\left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right). \tag{1.57}$$

Due to the complexity, we do not write down the explicit expression of the Kerr metric, but one can simply construct it following (1.49).

The Kerr-Schild double copy is a particular case of the *Weyl double copy*, which applies to all vacuum type D solutions and non-twisting type N solutions [52, 53].[4] The Weyl double copy is formulated in spinorial space. Any rank-k tensor can be translated into a rank-$2k$ spinorial object by employing the symbols $\sigma^\mu_{AA'}$

$$v_\mu \quad \to \quad v_{AA'} = v_\mu \sigma^\mu_{AA'}, \tag{1.58}$$

with $A, A' = 1, 2$ the spinor indices. In the Weyl representation, $\sigma^\mu_{AA'}$ can be written as

$$\sigma^a = \frac{1}{\sqrt{2}}\left(\mathbb{1}_{2\times 2}, \sigma^i_{2\times 2}\right), \tag{1.59}$$

where σ^i for $i = 1, 2, 3$ are the Pauli matrices. The spinor indices may be raised or lowered by the 2-dimensional Levi-Civita symbol,

$$\begin{gathered} v_A = \epsilon_{AB} v^B, \quad v^B = v_A \epsilon^{AB} \\ \text{with} \quad \epsilon_{AC}\epsilon^{BC} = \delta^B_A, \quad \text{and} \quad \epsilon_{01} = 1. \end{gathered} \tag{1.60}$$

The bispinor corresponding to the 1-form v_μ is explicitly

$$v_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} v_0 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & v_0 - v_3 \end{pmatrix}. \tag{1.61}$$

For higher-rank tensors, the conversion is similar but introduces more σ^μ symbols to contract with all Lorentz indices.

Let us now consider the field strength tensor $F_{\mu\nu}$ in Maxwell's theory, which is the Abelian version of Yang-Mills field. In the spinorial representation, it corresponds to

$$F_{\mu\nu} \quad \to \quad F_{AA'BB'} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB}. \tag{1.62}$$

The decomposition on the right-hand side is due to the anti-symmetry of $F_{\mu\nu}$. The symmetric quantities ϕ_{AB} and $\bar{\phi}_{A'B'}$ are the anti-self-dual and self-dual parts, respectively. Maxwell's equations are translated to

$$\begin{aligned} \nabla^{AA'}\phi_{AB} &= 0, \\ \nabla^{AA'}\bar{\phi}_{A'B'} &= 0, \end{aligned} \tag{1.63}$$

[4]The Petrov classification specifies the Weyl tensor at a given event based on the degeneracy of its eigenbivectors (or equivalently, principal null directions). There are precisely six types in total. A type D Weyl tensor has two double principal null directions, and A type N tensor has one quadruple principal null direction.

where $\nabla^{AA'} = \nabla^{\mu}\sigma_u^{AA'}$ is the incarnation of the spacetime covariant derivative in the spinor representation.

It turns out that the quantity corresponding to the field strength tensor $F_{\mu\nu}$ on the gravity side is the Weyl tensor $C_{\mu\nu\rho\sigma}$. Its relation to the Riemann tensor $R_{\mu\nu\rho\sigma}$ is

$$C_{\mu\nu}{}^{\rho\sigma} = R_{\mu\nu}{}^{\rho\sigma} - 4S_{[\mu}^{[\rho}\delta_{\nu]}^{\sigma]}, \tag{1.64}$$

where S_{μ}^{ρ} is the Schouten tensor given by

$$S_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{6}R\, g_{\mu\nu}\right), \tag{1.65}$$

with R the Ricci scalar. Translated to spinorial space, the quantity corresponds to the Weyl tensor is

$$C_{\mu\nu\rho\sigma} \quad \rightarrow \quad \Psi_{ABCD}\, \epsilon_{A'B'}\, \epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\, \epsilon_{AB}\, \epsilon_{CD} \tag{1.66}$$

Similar to the field strength, the Ψ_{ABCD} and $\bar{\Psi}_{A'B'C'D'}$ are the anti-self-dual and self-dual parts of the Weyl tensor respectively. Ψ_{ABCD} is usually referred to as the Weyl spinor. The Bianchi identity of the Riemann tensor gives constraints on the Weyl spinor

$$\begin{aligned} \nabla^{AA'}\Psi_{ABCD} &= 0, \\ \nabla^{AA'}\bar{\Psi}_{A'B'C'D'} &= 0. \end{aligned} \tag{1.67}$$

Comparing (1.63) and (1.67), the similarity between the equations of motion at the electromagnetism and the gravity side is apparent.

The Weyl double copy states that from a Maxwell field strength ϕ_{AB}, we can construct a Weyl spinor

$$\begin{aligned} \Psi_{ABCD} &= \frac{1}{S}\phi_{(AB}\phi_{CD)}, \\ \bar{\Psi}_{A'B'C'D'} &= \frac{1}{S}\bar{\phi}_{(A'B'}\bar{\phi}_{C'D')}, \end{aligned} \tag{1.68}$$

where S is a scalar function playing the same role with the $\varphi(x)$ in the Kerr-Schild double copy. The Weyl tensors constructed this way are naturally of Petrov type D. One may also try to combine different Maxwell spinors ϕ_{AB} and $\tilde{\phi}_{CD}$ to obtain Weyl spinors of other types, but its general validity remains unclear.

1.6 Worldline formalism

The worldline formalism is commonly employed as an effective theory to describe the classical dynamics of some quantum field theory. The most famous case is the worldline description of massive astronomical objects interacting via gravity, which could be seen as a low-energy effective theory for the (unknown) UV-complete quantum gravity. It can also describe charged particles coupled to an electromagnetic field or a non-Abelian gauge field. Here we introduce the worldline formalism by using particles moving in a gravitational field as an example.

The action can be generally expressed as

$$S = S_{\text{EH}} + S_{\text{gf}} + \sum_i S_{\text{pm}}^{(i)}. \tag{1.69}$$

S_{EH} is the usual Einstein-Hilbert action (1.35) and S_{gf} is a Faddeev-Popov gauge-fixing term. Note that in (1.69) we have included multiple worldlines to allow for interactions. In general, a point-mass coupled to gravity can be written as a 1-dimensional worldline embedded in spacetime,

$$S_{\text{pm}} = -m \int \mathrm{d}\sigma + c_R \int \mathrm{d}\sigma R(x) + c_V \int R_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + \mathcal{O}(R, \dot{x}^\mu) \tag{1.70}$$

with $\mathrm{d}\sigma = \sqrt{g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}$ being the proper time along the worldline, and $\dot{x}^\mu(\sigma)$ denotes the velocity defined in terms of the proper time. We omit terms in higher orders of the curvature and derivatives of x, which account for the interactions that involve extended objects or spinning effects.

We expect the first term to give rise to the geodesic equation. To see this, let us first parametrize it with respect to an arbitrary parameter τ,

$$-m \int \mathrm{d}\sigma = -m \int \mathrm{d}\tau \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}. \tag{1.71}$$

The Euler-Lagrange equation yields

$$\begin{aligned} 0 =& - m \left(\frac{\partial}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial}{\partial \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}} \right) \sqrt{g_{\rho\nu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}} \\ =& - m \frac{\partial_\mu g_{\rho\nu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}{2\sqrt{g_{\sigma\delta}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\delta}{\mathrm{d}\tau}}} + m\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}{\sqrt{g_{\sigma\delta}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\delta}{\mathrm{d}\tau}}} \right). \end{aligned} \tag{1.72}$$

To simplify this expression we introduce the following change of variable

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = \sqrt{g_{\sigma\delta}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\delta}{\mathrm{d}\tau}}. \tag{1.73}$$

The equation of motion then reads

$$0 = m\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \right) - \frac{\partial_\mu g_{\rho\nu}}{2}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \right), \tag{1.74}$$

Since $m\,\mathrm{d}\lambda/\mathrm{d}\tau \neq 0$, we can drop this factor and get

$$\begin{aligned} 0 =& g_{\mu\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\lambda^2} + \frac{1}{2}\left(2\partial_\rho g_{\mu\nu} - \partial_\mu g_{\rho\nu} \right)\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \\ =& g_{\mu\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\lambda^2} + \frac{1}{2}\left(\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\rho\nu} \right)\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}, \end{aligned} \tag{1.75}$$

where in the last equal sign we have used the fact that the second term is symmetric in the indices ρ, ν. Identifying the Christoffel symbol

$$\Gamma^\alpha{}_{\rho\nu} = \frac{1}{2}g^{\alpha\mu}\left(\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\nu\rho} \right), \tag{1.76}$$

and multiplying the inverse metric $g^{\alpha\mu}$, we obtain

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha{}_{\rho\nu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0. \tag{1.77}$$

In fact, from (1.73) we can see that λ is the proper time, thus we confirm that the above equation derived from the first term of the worldline action (1.70) gives the geodesic equation.

The following two terms given with the Wilson coefficients $c_{V/R}$ in (1.70) will be simply set to zero, since they can be removed by a field redefinition of $h_{\mu\nu}$ when computing gauge-invariant observables [79]. We note that these two terms may play a role for gauge-dependent quantities, such as the Schwarzschild metric. After these simplifications, another thing we can do is to introduce an einbein $e(\tau)$ to the point mass action,

$$S_{\text{pm}} = -\frac{1}{2}\int \mathrm{d}\tau \left(\frac{1}{e} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + m^2 e\right), \tag{1.78}$$

where $\dot{x}^\mu := \mathrm{d}x^\mu/\mathrm{d}\tau$ is the relativistic velocity parametrized by τ. This action enjoys the advantage that calculations are drastically simplified due to the absence of the square root. It also displays the gauge freedom of reparametrization of the worldline $\tau \to \tilde{\tau}(\tau)$ with

$$\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \;\to\; \frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{\tau}}\frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}\tau}, \qquad e(\tau) \;\to\; \tilde{e}(\tilde{\tau})\frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}\tau}. \tag{1.79}$$

For massive point particles, the equivalence to the first term of (1.70) can be shown by solving the equation of motion for $e(\tau)$,

$$-\frac{1}{e^2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + m^2 = 0 \qquad \Rightarrow \qquad e = \frac{1}{m}\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \tag{1.80}$$

and plugging it back to the action. It is instead more convenient for us to exploit the reparametrization gauge invariance and set $e(\tau) = 1/m$, which implies

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 1, \tag{1.81}$$

and τ is just the proper time.

Another advantage of introducing the einbein is that now we can describe massless particle as well. In this case, we do not have proper time any more, but we can choose an affine parametrization where $e(\tau)$ is just a constant. For simplicity we can set $e(\tau) = 1$, but we note that $e(\tau)$ has mass dimension -1.

The worldline formalism will be employed in several chapters in this thesis. Here we just show the most basic case coupled to Einstein gravity. Chapter 5 will used it to compute N-body potential in the PM limit. In chapter 3, we will also consider worldlines in more general backgrounds including dilaton-gravity, bi-adjoint scalar field and Yang-Mills theory.

Chapter 2

Double Copy of Massive Scalar QCD

This chapter is based on the published article "Double copy of massive scalar QCD" [1], in collaboration with Prof. Dr. Jan Plefka and Dr. Tianheng Wang. We will adapt the conventions for the consistency of this thesis.

To generalize the double copy to include massive particles, we find it natural to start by investigating amplitudes with massive states. In [80, 81], the tree level amplitudes of QCD with N_f massive spin-$1/2$ quarks are computed up to next-to-leading order and double copied to a $(\text{QCD})^{\otimes 2}$ gravitational theory. In $D = 4$, it contains an axion, a dilaton, a graviton field, various massive scalars, and massive vectors. The relation of the field content can be expressed as

$$[\, A_\mu \oplus (\, N_f \times \Psi \,)\,]^{\otimes 2} = h_{\mu\nu} \oplus B_{\mu\nu} \oplus \phi \oplus (N_f \times [\phi \oplus V^\mu])\,. \tag{2.1}$$

The gravitational $(\text{QCD})^{\otimes 2}$ Lagrangian is then reconstructed up to the sixth order in scalar fields by matching a proposed ansatz to the double copy amplitude. The generated Lagrangian is complicated due to the highly non-trivial interactions between the massless and massive states. In particular, contact terms among massive fields are founded to be necessary. A related paper is [82], where they explore the double copy of massive matter with spin $s < 2$ in general dimensions.

Inspired by the aforementioned work, we choose to study the double copy of massive scalar QCD (SQCD) to avoid the complexity resulting from taking the square of the spinning degree of freedom. This can be seen by the simpler matter fields

$$[\, A_\mu \oplus (\, N_f \times \varphi \,)\,]^{\otimes 2} = h_{\mu\nu} \oplus B_{\mu\nu} \oplus \phi \oplus (N_f \times \varphi)\,. \tag{2.2}$$

We will then construct an emerging gravitational Lagrangian up to sixth order in the scalar contact terms by matching the amplitudes. We will also take a field redefinition of the massive field and write the interaction in an elegant all-order resummed form.

This chapter is organized as the following. We will set up the stage of massive scalar QCD and work out the needed amplitudes up to next-to-leading order in the coupling constant in section 2.1. In section 2.2, we will perform the double copy approach to produce the gravitational amplitudes. The resulting Lagrangian of the scalar-two-form-dilaton-gravity is extracted from the amplitudes and generalized to all orders in 2.3.

2.1 Amplitudes of Massive Scalar QCD

2.1.1 Basics of SQCD

We consider scalar fields living in the fundamental representations of a gauge group coupled to gluons. In principle, what is described in this chapter applies to arbitrary gauge group, but for simplicity, we will use $SU(N)$, which is the case for electroweak ($SU(2)$) and quantum chromodynamics ($SU(3)$) in the standard model. The full Lagrangian reads

$$\mathcal{L}_{\rm SQCD} = \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm gh} + \mathcal{L}_{\rm scalar}, \tag{2.3}$$

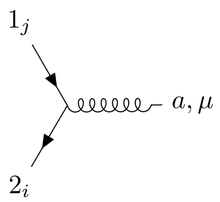
where $\mathcal{L}_{\rm YM}$ is the standard Yang-Mills Lagrangian (1.1), and we choose the Feynman gauge as the previous chapter (1.22). $\mathcal{L}_{\rm scalar}$ is the massive scalar Lagrangian,

$$\mathcal{L}_{\rm scalar} = \sum_{\alpha} \Big((D_\mu \varphi_{\alpha,i})^\dagger D^\mu \varphi_{\alpha,i} - m_\alpha^2 \varphi_{\alpha,i}^\dagger \varphi_{\alpha,i} \Big), \tag{2.4}$$

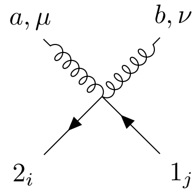
where $\alpha = 1, 2, \ldots, N_s$ labels the flavor of the massive scalar fields and i denotes the color index. We note that there are no contact interactions of massive scalar fields in this Lagrangian. The covariant derivative reads

$$D_\mu \varphi_{\alpha,i} = \partial_\mu \varphi_{\alpha,i} - igA_\mu^a T_{ij}^a \varphi_{\alpha,j}. \tag{2.5}$$

We can now derive the Feynman rules needed for amplitudes up to the sixth order in the scalar fields. The pure gluon Feynman rules are given in (1.23), (1.24) and (1.25). From the scalar Lagrangian (2.4), the couplings of massive scalars with gluons are read off



$$= igT_{ij}^a (p_1 - p_2)^\mu, \tag{2.6}$$



$$= ig^2 (T_{ik}^a T_{kj}^b + T_{ik}^b T_{kj}^a) \eta^{\mu\nu}. \tag{2.7}$$

Throughout this chapter, external momenta of scalars are taken as *incoming.*

2.1.2 SQCD Amplitudes

We are now ready to compute the SQCD amplitudes. Since it is well known that the double copy of pure Yang-Mills theory gives the two-form-dilaton-gravity ($\mathcal{N} = 0$ supergravity), we will focus on amplitudes involving at least one pair of external scalars. Therefore, the simplest amplitude is the 3-point amplitude of two massive scalars and one gluon

$$A(\underline{1}_{\alpha,j}, \overline{2}_{\alpha,i}, 3^a) = igT_{ij}^a \, \epsilon_3 \cdot (p_1 - p_2), \tag{2.8}$$

where the massive scalar particle and antiparticle are indicated by the under- and over-lines, respectively, and p_i denotes the external momentum of particle i.

The four-scalar amplitudes can also be straightforwardly obtained. When the two scalar pairs are of different flavors/masses, the amplitude reads

$$A(\underline{1}_{\alpha,j},\overline{2}_{\alpha,i},\underline{3}_{\beta,l},\overline{4}_{\beta,k}) = ig^2 T^a_{ij}T^a_{lk}\frac{2\,p_3\cdot(p_1-p_2)}{s_{34}}. \tag{2.9}$$

In this chapter, for amplitudes and Feynman diagrams with more than one pair of massive scalars, we use different colors to denote different flavors. For those with only one scalar-pair, there is no need to color it. Note that we use $s_{i..j} := (p_i + ... + p_j)^2$ throughout this section. When the two pairs of scalars are of the same flavor/mass, the amplitude is obtained from (2.9) by adding contributions from exchanging particles 2 and 4.

$$A(\underline{1}_{\alpha,j},\overline{2}_{\alpha,i},\underline{3}_{\alpha,l},\overline{4}_{\alpha,k}) = ig^2\left(T^a_{ij}T^a_{lk}\frac{2\,p_3\cdot(p_1-p_2)}{s_{34}} + T^a_{il}T^a_{kj}\frac{2\,p_3\cdot(p_1-p_4)}{s_{14}}\right). \tag{2.10}$$

In these 4-scalar amplitudes, there are not enough color structures to form Jacobi identities. The amplitude of two gluons and two scalars is slightly more involved but still straightforward,

$$A(\underline{1}_{\alpha,j},\overline{2}_{\alpha,i},3^a,4^b) = ig^2\left(\frac{(T^aT^b)_{ij}n_u}{s_{14}-m_\alpha^2} + \frac{(T^bT^a)_{ij}n_t}{s_{13}-m_\alpha^2} + \frac{f^{abc}T^c_{ij}n_s}{s_{34}}\right). \tag{2.11}$$

The kinematic numerators are given by

$$\begin{aligned} n_u &= [4(\epsilon_4\cdot p_1)(\epsilon_3\cdot p_2) + 2(\epsilon_3\cdot\epsilon_4)(p_1\cdot p_4)] \\ n_t &= [4(\epsilon_3\cdot p_1)(\epsilon_4\cdot p_2) + 2(\epsilon_3\cdot\epsilon_4)(p_1\cdot p_3)] \\ n_s &= [4(\epsilon_4\cdot p_1)(\epsilon_3\cdot p_2) - 4(\epsilon_3\cdot p_1)(\epsilon_4\cdot p_2) + 2(\epsilon_3\cdot\epsilon_4)\,p_1\cdot(p_4-p_3)]. \end{aligned} \tag{2.12}$$

We notice that the color and the kinematic numerators above automatically satisfy the relations below

$$(T^aT^b)_{ij} - (T^bT^a)_{ij} = f^{abc}T^c_{ij}, \tag{2.13}$$
$$n_u - n_t = n_s. \tag{2.14}$$

Hence, the kinematic numerators are in the BCJ-respecting representation and are ready to be squared in the double copy construction.

Proceeding to the six-point amplitude, we notice that the simplest case is the one with three external scalar pairs of distinct flavors/masses. The three topologies of the diagrams that contribute to this amplitude are given in the upper row of figure 2.1. In this case, the curly lines should be interpreted as gluons. After all the diagrams are summed up, the amplitude reads

$$\begin{aligned} A(\underline{1}_{\alpha,j},\overline{2}_{\alpha,i},\underline{3}_{\beta,l},\overline{4}_{\beta,k},\underline{5}_{\kappa,n},\overline{6}_{\kappa,m}) &= ig^4\frac{c_0n_0}{s_{12}s_{34}s_{56}} + ig^4\Bigg[\frac{c_{134}n_{134}}{(s_{134}-m_\alpha^2)\,s_{34}s_{56}} \\ &+ \frac{c_{156}n_{156}}{(s_{156}-m_\alpha^2)\,s_{34}s_{56}} + \big(\text{cyclic}\,[(1,2,\alpha)\to(3,4,\beta)\to(5,6,\kappa)]\,\big)\Bigg]. \end{aligned} \tag{2.15}$$

where the color factors are abridged to

$$\begin{aligned} c_0 &= T^a_{ij}T^b_{kl}T^c_{mn}f^{abc}, \\ c_{134} &= T^b_{ih}T^c_{hj}T^b_{kl}T^c_{mn}, \end{aligned} \tag{2.16}$$

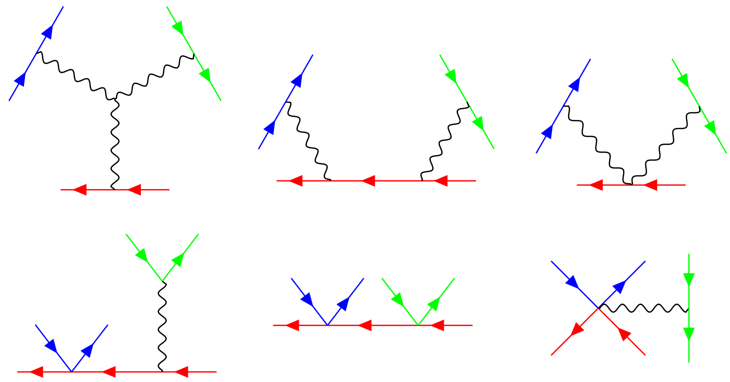


Figure 2.1: Topologies of Feynman diagrams of the 6-scalar amplitude. Curly lines denotes possible massless propagators.

$$c_{156} = T^c_{ih} T^b_{hj} T^b_{kl} T^c_{mn},$$

and the kinematic numerators read

$$\begin{aligned}
n_0 &= (p_1 - p_2)_\mu (p_3 - p_4)_\nu (p_5 - p_6)_\rho V^{\mu\nu\rho}_{p_1+p_2, p_3+p_4, p_5+p_6}, \\
n_{134} &= 4\, p_1 \cdot (p_3 - p_4)\, p_2 \cdot (p_5 - p_6) + (s_{134} - m_\alpha^2)\, (p_3 - p_4) \cdot (p_5 - p_6), \\
n_{156} &= 4\, p_1 \cdot (p_5 - p_6)\, p_2 \cdot (p_3 - p_4) + (s_{156} - m_\alpha^2)\, (p_3 - p_4) \cdot (p_5 - p_6).
\end{aligned} \tag{2.17}$$

The other numerators are gained from cyclic rotations as defined in (2.15). We have manipulated the kinematic numerators to be dual to the color factors,

$$c_0 = c_{134} - c_{156}\,, \qquad n_0 = n_{134} - n_{156}\,. \tag{2.18}$$

For the cases where the flavors are not all distinguished, more diagrams and the symmetry factors associated with the diagrams need to be taken care of. In the end, the additional contributions are obtained by permutations of $(1 \to 3 \to 5)$, similarly to the four-point amplitude. The other numerators are likewise constructed.

2.2 The Double Copy of Massive Scalar QCD Amplitudes

We then follow the standard BCJ procedure described in chapter 1 to perform the double copy procedure of the amplitudes. We first identify the degrees of freedom of the resulting theory. Their associated polarization tensors are identified with the tensor products of the gluon polarization vectors in the following ways,

$$\text{graviton} \quad : \quad (\epsilon^h)^{ij}_{\mu\nu} = \epsilon^{((i}_\mu \epsilon^{j))}_\nu, \tag{2.19}$$

$$\text{B-field} \quad : \quad (\epsilon^B)^{ij}_{\mu\nu} = \epsilon^{[i}_\mu \epsilon^{j]}_\nu, \tag{2.20}$$

$$\text{dilaton} \quad : \quad (\epsilon^\phi)_{\mu\nu} = \frac{\epsilon^i_\mu \epsilon^j_\nu \delta_{ij}}{\sqrt{D-2}}, \tag{2.21}$$

where the double parenthesis denotes taking the symmetric-traceless part of the tensor product, and the square bracket refers to anti-symmetrization. Note that the superscripts $i, j = 1, 2, \ldots (D-2)$ are not related to the color group as in (2.4), but are the little group indices.[1] The dilaton accounts for the trace of the tensor product hence the i, j indices are contracted with δ_{ij}[2]. It is written as a rank-2 tensor state in (2.21), although we know that the dilaton is actually a scalar. To illustrate this, we can further simplify the identification as

$$(\epsilon^{\phi})_{\mu\nu}(p,q) = \frac{1}{\sqrt{D-2}}\left(-\eta_{\mu\nu} + \frac{p_\mu q_\nu + p_\nu q_\mu}{p\cdot q}\right), \tag{2.22}$$

where p is the momentum associated with the external particle, and q is an arbitrary reference null-vector. We see that there is no dependence on the polarization vectors.

Focusing only on the scattering processes that involve at least one pair of massive scalars, we now present all the relevant double copy results up to 4-point as well as the 6-scalar amplitude. The simplest case on the YM side is the 3-point amplitude (2.8) with two scalars and one gluon. Its double copy gives two different amplitudes,

$$\mathcal{M}(\underline{1},\overline{2},3_h) = i\kappa(\epsilon_3^h)_{\mu\nu}p_1^\mu p_2^\nu, \tag{2.23}$$

$$\mathcal{M}(\underline{1},\overline{2},3_\phi) = \frac{i\kappa m^2}{\sqrt{D-2}}. \tag{2.24}$$

Because of our choice of conventions (1.4) and (1.12), the relation between the coupling constants is $ig \to -i\kappa/4$ in the double copy procedure. The amplitude vanishes when the polarization is anti-symmetrized, so the external massless state cannot be a B-field. This applies to any amplitude that involves an odd number of B-fields. Therefore we will only write down the non-vanishing ones.

Of the 4-point amplitudes with two massive scalar and two gluons, the double copy gives richer outcomes. All the distinct resulting amplitudes are

$$\begin{aligned}\mathcal{M}(\underline{1},\overline{2},3_h,4_h) =& i\kappa^2(\epsilon_3^h)_{\mu\nu}(\epsilon_4^h)_{\rho\sigma}\Bigg[\frac{1}{s_{34}}\Big(s_{13}p_1^\rho p_2^\mu\eta^{\nu\sigma} + s_{14}p_1^\mu p_2^\rho\eta^{\nu\sigma} - p_1^\mu p_1^\nu p_2^\rho p_2^\sigma - p_1^\rho p_1^\sigma p_2^\mu p_2^\nu \\ &+ 2p_1^\mu p_1^\rho p_2^\nu p_2^\sigma + \frac{1}{4}s_{13}s_{14}\eta^{\mu\rho}\eta^{\nu\sigma}\Big) - \frac{p_1^\mu p_1^\nu p_2^\rho p_2^\sigma}{s_{13}-m^2} - \frac{p_1^\rho p_1^\sigma p_2^\mu p_2^\nu}{s_{14}-m^2}\Bigg],\end{aligned} \tag{2.25}$$

$$\mathcal{M}(\underline{1},\overline{2},3_\phi,4_h) = \frac{i\kappa^2 m^2(\epsilon_4^h)_{\mu\nu}}{\sqrt{D-2}}\left(\frac{p_3^\mu p_3^\nu}{s_{34}} + \frac{p_1^\mu p_1^\nu}{s_{14}-m^2} + \frac{p_2^\mu p_2^\nu}{s_{13}-m^2}\right), \tag{2.26}$$

$$\mathcal{M}(\underline{1},\overline{2},3_\phi,4_\phi) = \frac{i\kappa^2(p_1\cdot p_3)(p_1\cdot p_4)}{s_{34}} - \frac{i\kappa^2}{D-2}\left(\frac{m^4}{s_{13}-m^2} + \frac{m^4}{s_{14}-m^2} + m^2\right), \tag{2.27}$$

$$\begin{aligned}\mathcal{M}(\underline{1},\overline{2},3_B,4_B) =& \frac{i\kappa^2}{s_{34}}(\epsilon_3^B)_{\mu\nu}(\epsilon_4^B)_{\rho\sigma}\Big(2p_1^\mu p_2^\nu p_1^\rho p_2^\sigma - 2(p_1\cdot p_3)p_1^\rho p_2^\mu\eta^{\nu\sigma} \\ &- 2(p_1\cdot p_4)p_1^\mu p_2^\rho\eta^{\nu\sigma} - (p_1\cdot p_3)(p_1\cdot p_4)\eta^{\mu\rho}\eta^{\nu\sigma}\Big).\end{aligned} \tag{2.28}$$

The SQCD 4-scalar amplitudes, with the two pairs of particles being of distinct flavor, (2.9) is double-copied to

$$\mathcal{M}(\underline{1}_\alpha,\overline{2}_\alpha,\underline{3}_\beta,\overline{4}_\beta) = -i\kappa^2\left(\frac{s_{34}}{16} - \frac{p_1\cdot p_3\; p_2\cdot p_3}{s_{34}}\right). \tag{2.29}$$

[1] In Wigner's classification of particles, a little group is a subgroup of the Poincaré group which leave the momentum invariant. In four-dimensional Minkowskian spacetime, it's $SO(3)$ and $E(2)$ for massive and massless particles, respectively.

[2] We assume that the polarization vectors are on a Cartesian basis. In other bases, δ_{ij} should be expressed in that specific basis accordingly. For example, in $D=4$, in terms of the helicity eigenstates, the external dilaton is $(e_\mu^+ e_\nu^- + e_\mu^- e_\nu^+)/\sqrt{2}$.

Similarly, in the case where the particles are of the same flavour, the double copy of the 4-scalar amplitude (2.10) yields

$$\mathcal{M}(\underline{1}_\alpha, \overline{2}_\alpha, \underline{3}_\alpha, \overline{4}_\alpha) = -i\kappa^2\left(\frac{s_{23}+s_{34}}{16} - \frac{p_1\cdot p_3\ p_2\cdot p_3}{s_{34}} - \frac{p_1\cdot p_3\ p_3\cdot p_4}{s_{23}}\right). \tag{2.30}$$

The 6-scalar amplitude is also directly gained from (2.15),

$$\begin{aligned}\mathcal{M}(\underline{1}_{\alpha,j}, \overline{2}_{\alpha,i}, \underline{3}_{\beta,l}, \overline{4}_{\beta,k}, \underline{5}_{\kappa,n}, \overline{6}_{\kappa,m}) = i\frac{\kappa^4}{256}\frac{n_0^2}{s_{12}s_{34}s_{56}} + i\frac{\kappa^4}{256}\Bigg[\frac{n_{134}^2}{(s_{134}-m_\alpha^2)\,s_{34}s_{56}} \\ + \frac{n_{156}^2}{(s_{156}-m_\alpha^2)\,s_{34}s_{56}} + \big(\text{cyclic}\,[(1,2,\alpha)\to(3,4,\beta)\to(5,6,\kappa)]\,\big)\Bigg],\end{aligned} \tag{2.31}$$

where the numerators are again given by (2.17).

2.3 Two-Form-Dilaton-Gravity with Massive Scalars

2.3.1 Action and Feynman Rules

Our starting point is the Lagrangian of the two-form-dilaton-gravity,

$$\mathcal{L}_{\text{adg}} = \sqrt{|g|}\Big[-\frac{2}{\kappa^2}R + \frac{D-2}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{6}e^{-2\kappa\phi}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\Big], \tag{2.32}$$

where $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$ is the field-strength associated to the two-form and R the Ricci scalar. The interactions involving massive scalars may now be extracted from the amplitudes computed above. In fact, we found that the minimal coupling of massive scalars and gravity reproduces the 2-massive-1-graviton amplitude (2.23) and the 2-massive-2-graviton amplitude (2.25). It is straightforward to extend the graviton couplings to higher orders,

$$\mathcal{L}_{\text{matter}} = \sqrt{|g|}\sum_\alpha\Big(g^{\mu\nu}\partial_\mu\varphi_\alpha^\dagger\partial_\nu\varphi_\alpha - m_\alpha^2\varphi_\alpha^\dagger\varphi_\alpha\Big)\,. \tag{2.33}$$

From the 4-point order, we can get all other interaction terms up to quadratic order in κ. The 6-scalar amplitude gives us the 6-scalar contact term. In the end, we find the following Lagrangian from the double copy

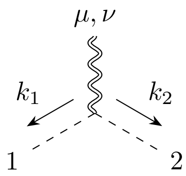
$$\begin{aligned}\mathcal{L}_{\text{DC}} = \mathcal{L}_{\text{adg}} + \sqrt{|g|}\sum_\alpha\Bigg[g^{\mu\nu}\partial_\mu\varphi_\alpha^\dagger\partial_\nu\varphi_\alpha - m_\alpha^2 e^{-\kappa\phi}\varphi_\alpha^\dagger\varphi_\alpha \\ + \Big(\frac{\kappa^2}{32}\varphi_\alpha^\dagger\varphi_\alpha - \frac{\kappa^4}{512}(\varphi_\alpha^\dagger\varphi_\alpha)^2\Big)D_\mu D^\mu\Big(\sum_\beta\varphi_\beta^\dagger\varphi_\beta\Big)\Bigg],\end{aligned} \tag{2.34}$$

where D_μ is the covariant derivative with the normal Levi-Civita connection. It is required by the covariance of the Lagrangian. The dilaton propagator from (2.32) is

$$\text{- - - - - - - -} = \frac{i}{(D-2)p^2}, \tag{2.35}$$

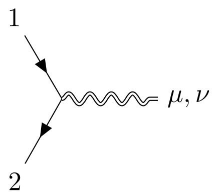
The graviton propagator and pure-graviton vertices are given in section 1.3. We do not need the propagator of the B-field. Since the kinetic term of dilaton is non-canonical, we also need to dress each external dilaton by a factor $1/\sqrt{D-2}$. The dimension dependence of the propagators of dilaton and graviton will cancel each other in pure massive scalar amplitudes as expected from the SQCD side.

As introduced in chapter 1, in the weak-field approximation, we expand the metric as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. This yields the couplings of graviton and dilatons,

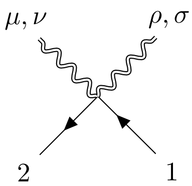


$$= -i\kappa\frac{D-2}{2}\left(k_1\cdot k_2\,\eta^{\mu\nu} - 2\,k_1^{\mu}k_2^{\nu}\right), \tag{2.36}$$

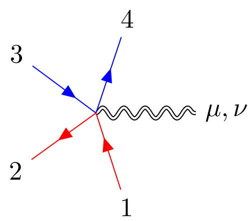
as well as the vertices of gravitons coupled to massive scalars are



$$= -i\kappa\left(\frac{\eta^{\mu\nu}}{2}(p_1\cdot p_2 + m^2) - p_1^{(\mu}p_2^{\nu)}\right), \tag{2.37}$$

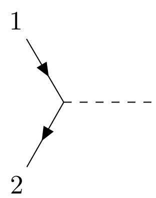


$$\begin{aligned} &= \frac{i\kappa^2}{2}\Big[\eta^{\rho\sigma}p_1^{(\nu}p_2^{\mu)} + \eta^{\mu\nu}p_1^{(\sigma}p_2^{\rho)} - 2p_1^{(\sigma}\eta^{\rho)(\mu}p_2^{\nu)} - 2p_1^{(\mu}\eta^{\nu)(\sigma}p_2^{\rho)} \\ &\qquad + \Big(m^2 + p_1\cdot p_2\Big)\Big(\eta^{\mu(\rho}\eta^{\sigma)\nu} - \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\Big)\Big], \end{aligned} \tag{2.38}$$

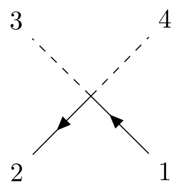


$$= \frac{i\kappa^3}{32}\Big[(p_1+p_2)\cdot(p_3+p_4)\eta^{\mu\nu} - 2(p_1^{\mu}+p_2^{\mu})(p_3^{\nu}+p_4^{\nu})\Big]. \tag{2.39}$$

Note that the last vertex involves two different pairs of scalars in blue and red respectively. For the vertex with two pairs of identical scalars and a graviton, we simply take (2.39) and add another contribution by exchanging labels 2 and 4. The vertices of massive scalars coupled to one or two gravitons deduced from (2.34) are,

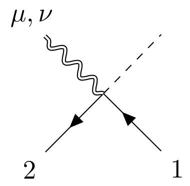


$$= i\kappa m^2, \tag{2.40}$$



$$= -i\kappa^2 m^2. \tag{2.41}$$

The four-point vertex of a pair of massive scalar, a dilaton and a graviton is also derived from (2.34),



$$= \frac{i\kappa^2 m^2 \eta_{\mu\nu}}{2}. \tag{2.42}$$

2.3.2 Matching to Double Copy Amplitude

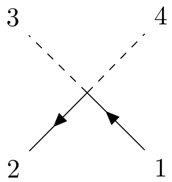
Let us now explain the extraction of each contact term in (2.34). We first make an ansatz on how the dilaton couples to massive scalars. It is also known that the dilaton appears as an exponent when coupled to other fields,

$$\mathcal{L}_{\phi\varphi^\dagger\varphi} = \sqrt{|g|}\left(e^{\lambda\kappa\phi} g^{\mu\nu}\partial_\mu\varphi_\alpha^\dagger\partial_\nu\varphi_\alpha - m_\alpha^2 e^{\zeta\kappa\phi}\varphi_\alpha^\dagger\varphi_\alpha\right), \tag{2.43}$$

whose leading term in κ gives the canonical kinetic and mass terms. We can compute the 3-point amplitude of two matter fields and a dilaton,

$$\mathcal{M}(\underline{1}, \overline{2}, 3_\phi) = \frac{i(\lambda - \zeta)\kappa m^2}{\sqrt{D-2}}. \tag{2.44}$$

Matching to the double copy result (2.24), we have $\lambda - \zeta = 1$. The 2-massive-2-dilaton amplitude from double copy (2.27) subtracted by all Feynman diagrams that contain only cubic vertices yields the diagram that comes from the contact term,



$$= \frac{-i\kappa^2}{2}\left(m^2(1 + 2\lambda\zeta - 2\lambda^2) + \frac{\lambda^2}{4}s_{34}\right). \tag{2.45}$$

We can also calculate this diagram directly from Feynman rules of the 4-point contact term,

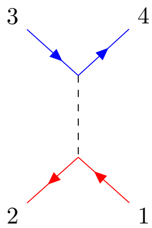
$$\frac{-i\kappa^2}{2}\left(m^2(\zeta^2 - \lambda^2) + \frac{\lambda^2}{2}s_{34}\right). \tag{2.46}$$

Comparing the above formulae, (2.45) and (2.46), we have two equations of λ and ζ. Together with $\lambda - \zeta = 1$, we find the solution,

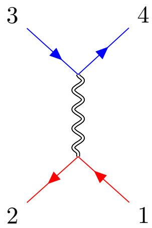
$$\lambda = 0, \qquad \zeta = -1. \tag{2.47}$$

We also verified that (2.43) gives the correct 4-point amplitude of 2 massive scalars, 1 dilaton and 1 graviton, of (2.26). As for the 2 massive scalar and 2 B-field scattering, we compute the amplitude from the perturbative Lagrangian (2.34). It coincides with (2.28), so there is no direct coupling of the B-field and massive scalars at least at 3- and 4-point levels.

We now proceed to consider the interactions among the massive scalars. These will be presented in more detail since it is this chapter's main result. The amplitude of 2 pairs of massive scalars of different flavors contains contributions from the following diagrams,

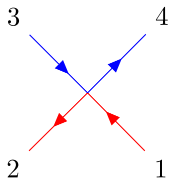


$$= \frac{-i\kappa^2 m_1^2 m_2^2}{(D-2)s_{34}}, \tag{2.48}$$



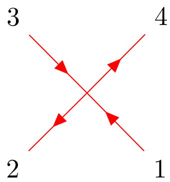
$$= i\kappa^2 \left(\frac{m_1^2 m_2^2}{(D-2)s_{34}} + \frac{p_1 \cdot p_3 \; p_2 \cdot p_3}{s_{34}} \right). \tag{2.49}$$

In order to match the double copy amplitude (2.29), a 4-scalar contact term is required,



$$= -i\kappa^2 \frac{s_{34}}{16}. \tag{2.50}$$

A similar calculation applies to the 4-scalar amplitude, where all massive scalars are of the same flavor. We only need to add the contribution by exchanging p_1 and p_3. The contact term will be

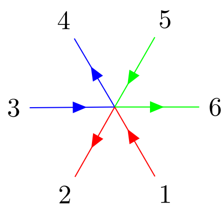


$$= -i\kappa^2 \frac{s_{34} + s_{14}}{16}. \tag{2.51}$$

In summary, the 4-scalar interaction term can be extracted as

$$\mathcal{L}_{\varphi^4} = \sqrt{|g|} \frac{\kappa^2}{32} \varphi_\alpha^\dagger \varphi_\alpha D_\mu D^\mu (\varphi_\beta^\dagger \varphi_\beta). \tag{2.52}$$

The 6-scalar contact term is computed in the same way. For simplicity, the 3 pairs of scalars are taken to be of different flavors. Cases where two or three pairs of scalar are of the same flavor can be gained by simply exchanging the external particles. We match the amplitude computed from summing all Feynman diagrams to the double copy result. All the diagrams, except for the one from the 6-scalar interaction, are categorized into the six structures in fig. 2.1. Subtracting every diagram that falls into either category in fig. 2.1 from the double copy amplitude will give us the 6-scalar contact term,



$$= \frac{i\kappa^4}{256} (s_{12} + s_{34} + s_{56}). \tag{2.53}$$

The corresponding interaction will be

$$\mathcal{L}_{\varphi^6} = -\sqrt{|g|}\frac{\kappa^4}{512}(\varphi_\alpha^\dagger\varphi_\alpha)^2 D_\mu D^\mu(\varphi_\beta^\dagger\varphi_\beta). \tag{2.54}$$

We now have calculated every term in (2.34).

2.4 Further details on the Double Copy Action

The established higher-order terms in the action (2.34) do look somewhat non-standard with the double derivative $D^\mu D_\mu$ appearing. We would now like to unify them with the kinetic and mass term in (2.34) by performing a suitable field redefinition.[3] To this end, we consider the shift

$$\varphi_\alpha \to \varphi_\alpha + \frac{\kappa^2}{32}\varphi_\alpha\,(\varphi_\beta^\dagger\varphi_\beta) + \frac{\kappa^4}{1024}\varphi_\alpha\,(\varphi_\beta^\dagger\varphi_\beta)^2 \tag{2.55}$$

that transforms the matter part of the Lagrangian (2.34) into

$$\mathcal{L}_{\text{DCmatter}} = \sqrt{|g|}\,\Big(g^{\mu\nu}\partial_\mu\varphi_\alpha^\dagger\partial_\nu\varphi_\alpha - m_\alpha^2 e^{-\kappa\phi}\varphi_\alpha^\dagger\varphi_\alpha\Big)\Big(1+\frac{\kappa^2}{16}(\varphi_\beta^\dagger\varphi_\beta)+\frac{3\kappa^4}{1024}(\varphi_\beta^\dagger\varphi_\beta)^2+\mathcal{O}(\kappa^6)\Big). \tag{2.56}$$

This action after the field redefinitions and the previous one (2.34) have identical scattering amplitudes[4]. This form of the action suggests itself to an attractive resummation into the compact form

$$\mathcal{L}_{\text{DC}} = \mathcal{L}_{\text{adg}} + \sqrt{|g|}\frac{\partial_\mu\varphi_\alpha^\dagger\partial^\mu\varphi_\alpha - m_\alpha^2 e^{-\kappa\phi}\varphi_\alpha^\dagger\varphi_\alpha}{\left(1-\frac{\kappa^2}{32}\varphi_\beta^\dagger\varphi_\beta\right)^2}\,, \tag{2.57}$$

with $\mathcal{L}_{\text{adg}}$ the two-form-dilaton-gravity theory of (2.32). Note that in 4D upon dualizing the two-form to an axion χ via $d\chi = \frac{4}{\kappa}\,e^{2\kappa\,\phi}\,*\,dB$ the axio-dilaton system displays a striking similarity to the massive flavored scalar Lagrangian above. In 4D the double copy of scalar QCD takes the form

$$\mathcal{L}_{(\text{SQCD})^2} = -\frac{2\sqrt{|g|}}{\kappa^2}R + \sqrt{|g|}\frac{\partial_\mu\bar{Z}\partial^\mu Z}{(1-\frac{\kappa^2}{4}\bar{Z}Z)^2} + \sqrt{|g|}\frac{\partial_\mu\varphi_\alpha^\dagger\partial^\mu\varphi_\alpha - m_\alpha^2 e^{-\kappa\phi}\varphi_\alpha^\dagger\varphi_\alpha}{\left(1-\frac{\kappa^2}{32}\varphi_\beta^\dagger\varphi_\beta\right)^2}\,. \tag{2.58}$$

Here the complex scalar field Z is built from the dilaton ϕ and axion χ as

$$Z = \frac{2}{\kappa}\frac{\kappa\chi + i(e^{-\kappa\phi}-1)}{\kappa\chi + i(e^{-\kappa\phi}+1)} \tag{2.59}$$

enjoys an $\mathrm{SL}(2,\mathbf{R})$ symmetry (see e.g. [83]). We note in closing that such a symmetry is also present in the scalar sector for the *massless* $N_f = 1$ case.

In conclusion, we explicitly constructed the double copy of scalar QCD in arbitrary dimensions resulting in an extension of the established two-form-dilaton gravity model

[3] We thank Alexander Ochirov for crucial discussions on this point.

[4] For this the asymptotic equivalence of the two fields φ_α and $\varphi'_\alpha = \varphi_\alpha + \delta\varphi_\alpha$ is required, i.e. it suffices if $\delta\varphi_\alpha$ is higher order in fields (and couplings).

("$\mathcal{N}=0$ supergravity") by interacting massive flavored scalars displaying self-interactions to arbitrary quadratic field orders. These self-interaction terms are short-range contact terms and hence have no contributions to classical physics. Additionally, we found no coupling of the flavored scalars to the two-form, i.e. axion in 4 dimensions, as is commonly expected, yet not proven. We thus confirm that that at the classical level, it is sufficient to consider massive particles coupled minimally to the dilaton gravity as the resulting consistent theory of double copy of the classical version of scalar QCD, at least up to next-to-leading order.

The simple form of the proposed double copy Lagrangian (2.57) suggests that it might be constrained by some underlying symmetries. It is not clear to us how to find these possible constraints. To obtain a completed double copy theory, one has to go beyond perturbation theory as we did in this chapter. This is very challenging and to the best of our knowledge, this is no known method to do so at present.

Chapter 3

Classical Double Copy of Worldline QFT

This chapter is based o n the published article "Classical Double Copy of Worldline Quantum Field Theory" [2], written in collaboration with Prof. Dr. Jan Plefka.

Following the quantum story of the double copy of massive particles, we would like to explore the possibility of bypassing the quantum theory and directly performing double copy procedure at the classical level. The worldline formalism has been employed for years as an effective theory to describe classical massive particles coupled to gravity, such as black holes and neutron stars. Moreover, it is known that the quantization of worldlines describes quantum states in corresponding quantum field theory. Recently, Mogull et al. used a worldline quantum field theory to model classical gravitational scattering of two massive black holes. In WQFT, physical quantities are calculated as expectation values of corresponding operators rather than solving equations of motions, and it is found to be directly linked to the S-matrix elements in quantum theory of massive quarks. This connection suggests that at the classical level, it is expected that the double copy can be realized for physical quantities which are gauge-independent and on-shell akin to quantum scattering amplitudes. In practice, physical observables are efficiently computed by summing up all Feynman diagrams, which is another similarity to amplitudes.

The WQFT was originally developed to describe and simplify calculations of massive particles interacting via pure Einstein gravity. To investigate the classical double copy, we extend the formalism by considering massive (colored) point particles coupled to a bi-adjoint scalar, Yang-Mills field, and dilaton gravity. The bi-adjoint scalar theory is required to restore the locality structure, as explained in the following sections. We establish a prescription of a perturbative classical double copy of these three theories. We also illustrate the validity of the double copy by calculating the eikonal phase up to subleading order in all three theories. Their connection to scattering amplitudes is clear and is shown explicitly at next-to-leading order. For simplicity, we will focus on the $D = 4$ case in this chapter.

We will first briefly introduce the framework of worldline quantum field theory in section 3.1, focusing on three specific models where massive objects coupled to a bi-adjoint scalar, Yang-Mills field, and dilaton gravity, respectively. We will explain in detail how to calculate observables in WQFT and presents the necessary Feynman rules. It is followed by section 3.2, where we show the double copy of the eikonal phase up to subleading order in the coupling constants. The relation of radiation and the eikonal double copy is discussed in section 3.3. We further reveal the relationship of the WQFT and amplitude double copy, building upon the connection between eikonal approximation and the classical limit of scattering amplitude in section 3.4.

3.1 Basics of Worldline Quantum Field Theory

As an introduction to the basic ideas on WQFT, we first consider the theory of massive scalars coupled minimally to pure gravity. This theory successfully describes the scattering of two non-spinning black holes up to order $\mathcal{O}(G^2)$ [84–89].

We take the standard worldline-gravity action (1.69) and adopt the de Donder gauge (1.39). Setting the einbein as $e(\tau) = 1/m$, the worldline action (1.78) becomes

$$S_{\text{pm}} = -\frac{m}{2}\int \mathrm{d}\tau\,(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + 1)\,. \tag{3.1}$$

We expand the metric around Minkowskian spacetime as usual. Additionally, we consider the worldline to have a small deviation from a straight-line trajectory,

$$x^\mu(\tau) = b^\mu + v^\mu\tau + z^\mu(\tau), \tag{3.2}$$

where b^μ is a constant vector which will be later related to the impact parameter, v^μ is the velocity of the background straight-line orbit with $v^2 = 1$. We take $b \cdot v = 0$ which may always be achieved upon shifting τ. This straight-line expansion is only valid for unbound orbits, and we are thus limited to the scattering process. We note that it is consistent with the small momentum exchange limit, which is essentially equivalent to the classical limit of scattering amplitude. In principle, the background trajectory can be defined at arbitrary reference time τ_0 as long as the perturbation field satisfies

$$z^\mu(\tau_0) = 0, \qquad \dot{z}^\mu(\tau_0) = 0. \tag{3.3}$$

However, for our convenience, we usually choose to define it in the past infinity ($\tau_0 = -\infty$). In this case, the straight line coincides with the exact trajectory in the past infinity. Recently, it has been realized that this setting corresponds to the "in-in" formalism that naturally arises when one derives classical observables from scattering amplitudes [44]. This condition also induces us to use retarded propagators to compute physical quantities.

In WQFT, we treat the perturbation of the metric and the worldline with an equal footing. Practically, we will integrate out the fluctuations $h_{\mu\nu}$ and z^μ with the path integral formalism to arrive at the observables that depend only on the background parameters. As stated before, physical quantities are computed as the expectation values of the corresponding operators. In the path integral, we can express the expectation value of an operator $\mathcal{O}$ as

$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}_{\text{WQFT}}}\int D[h_{\mu\nu}]\prod_i D[z_i]\mathcal{O}\, e^{iS_{\text{WQFT}}}, \tag{3.4}$$

where we have included multiple numbers of worldlines denoted by index i. $\mathcal{Z}_{\text{WQFT}}$ is the partition function,

$$\mathcal{Z}_{\text{WQFT}} = \int D[h_{\mu\nu}]\prod_i D[z_i]e^{iS_{\text{WQFT}}}. \tag{3.5}$$

In the binary case ($i = 1, 2$), it turns out that the eikonal phase χ may be identified with the logarithm of the partition function,

$$\chi = -i \ln \mathcal{Z}_{\text{WQFT}}. \tag{3.6}$$

The momentum deflection of a particle Δp_i^μ can be calculated by taking the derivative of the eikonal with respect to the impact parameter b_i^μ. Here we claim that this relation holds for an arbitrary number of worldlines. The proof is easily done in the path integral. We consider the derivative of $\ln \mathcal{Z}_{\text{WQFT}}$ with respect to b_i^μ,

$$i\frac{\partial \ln \mathcal{Z}_{\text{WQFT}}}{\partial b_i^\mu} = \left\langle -\frac{\partial S_{\text{WQFT}}}{\partial b_i^\mu} \right\rangle = -\int_{-\infty}^{+\infty} \mathrm{d}\tau \left\langle \frac{\partial L_{\text{pm}}}{\partial x_i^\mu} \right\rangle, \tag{3.7}$$

where in the last step we exploit the fact that in the full action b_i^μ only appears as the τ-independent background of $x_i^\mu(\tau)$ in the point particle action $S_{\text{pm}} = \int \mathrm{d}\tau L_{\text{pm}}$, where L_{pm} is the Lagrangian. As the expectation value of the equation of motion for $x(\tau)$ vanishes, using the Euler-Lagrange equations, we can rewrite eq. (3.7) as

$$i\frac{\partial \ln \mathcal{Z}_{\text{WQFT}}}{\partial b_i^\mu} = -\int_{-\infty}^{+\infty} \mathrm{d}\tau \left\langle \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial L_{\text{pm}}}{\partial \dot{x}_i^\mu} \right\rangle = \langle p_{i,\mu}^{\text{ca}} \rangle \big|_{-\infty}^{+\infty}, \tag{3.8}$$

where $p_{i,\mu}^{\text{ca}} = -\partial L_{\text{pm}} / \partial \dot{x}_i^\mu$ is the canonical momentum conjugated to x^μ. Since we are studying a scattering process, we may assume that the point particles are so far separated that the interaction terms vanish in the past and future infinity. In this case $p_{i,\mu}^{\text{ca}}$ reduces to the kinematic momentum $m_i \dot{x}_i^\mu$, so we have

$$m_i \Delta \dot{x}_i^\mu = i\frac{\partial \ln \mathcal{Z}_{\text{WQFT}}}{\partial b_{i,\mu}}. \tag{3.9}$$

We therefore conclude that the eikonal is equivalent to the partition function in the sense that they both derive the momentum deflection by taking the derivative with respect to the impact parameter.

In practice, we compute the Feynman diagrams contributing to the expectation values. The Feynman rule for fields in the bulk is the same as in pure gravity. To extract the propagator of the worldline fluctuation $z(\tau)$, we plug the weak field and worldline expansion (3.2) into (3.1). The terms independent of $h_{\mu\nu}$ reads

$$S_{\text{pm}}^{(0)} = -\frac{m}{2} \int \mathrm{d}\tau \, \dot{x}^2 = -\frac{m}{2} \int \mathrm{d}\tau \left(2 + 2v \cdot \dot{z} + \dot{z}^2 \right). \tag{3.10}$$

The first term in the bracket is just a constant, and the second is a boundary term, so they are negligible. The third term is the kinetic term of $z(\tau)$, from which we can read off the propagator

$$z^\mu \xrightarrow{\omega} z^\nu = -\frac{i}{m} \frac{\eta^{\mu\nu}}{\omega^2}. \tag{3.11}$$

Since we are in the classical regime, we only care about tree diagrams and no ghosts are needed. It is more natural to work in the momentum space to describe the interaction of graviton and deflection,

$$\begin{aligned} h_{\mu\nu}(x) &= \int_k e^{ik\cdot x} h_{\mu\nu}(-k), \\ z^\mu(\tau) &= \int_\omega e^{-i\omega\tau} z^\mu(\omega). \end{aligned} \tag{3.12}$$

For convenience, we use the integral shorthands,

$$\begin{aligned} \int_\omega := \int \frac{\mathrm{d}\omega}{2\pi}, \qquad \int_k := \int \frac{\mathrm{d}^4 k}{(2\pi)^4}, \\ \bar{\delta}(\omega) := 2\pi\delta(\omega), \qquad \bar{\delta}^{(4)}(k^\mu) := (2\pi)^4\delta^{(4)}(k^\mu) \end{aligned} \tag{3.13}$$

to avoid proliferation of the 2π factors. We will leave the calculation details when we talk about the specific models we need in this chapter. Since the deviation from the straight line is implicitly in higher-order in G, when evaluated on the worldline, the graviton field can be further expanded,

$$\begin{aligned} h_{\mu\nu}(x(\tau)) &= \int_k e^{ik\cdot(b+v\tau+z(\tau))} h_{\mu\nu}(-k) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_k e^{ik\cdot(b+v\tau)} (k\cdot z(\tau))^n h_{\mu\nu}(-k) \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{k,\omega_1,\ldots,\omega_n} e^{ik\cdot b} e^{i\left(k\cdot v + \sum_{i=1}^n \omega_i\right)\tau} \left(\prod_{i=1}^n k\cdot z\left(-\omega_i\right)\right) h_{\mu\nu}(-k). \end{aligned} \tag{3.14}$$

This produces infinitely many linear interactions in $h_{\mu\nu}$, collectively written as

$$\begin{aligned} S_{\mathrm{pm}}^{\mathrm{int}} =& S_{\mathrm{pm}} - S_{\mathrm{pm}}^{(0)} = -\kappa m \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{k,\omega_1,\ldots,\omega_n} e^{ik\cdot b} \bar{\delta}\Big(k\cdot v + \sum_{i=1}^n \omega_i\Big) h_{\mu\nu}(-k) \Big(\prod_{i=1}^n z^{\rho_i}(-\omega_i)\Big) \\ & \times \left[\frac{1}{2}\Big(\prod_{i=1}^n k_{\rho_i}\Big) v^\mu v^\nu + \sum_{i=1}^n \omega_i \Big(\prod_{j\neq i}^n k_{\rho_j}\Big) v^{(\mu}\delta^{\nu)}_{\rho_i} + \sum_{i<j}^n \omega_i\omega_j \Big(\prod_{l\neq i,j}^n k_{\rho_l}\Big) \delta^{(\mu}_{\rho_i}\delta^{\nu)}_{\rho_j}\right], \end{aligned} \tag{3.15}$$

where we have integrated out τ to bring it fully to momentum space.

3.1.1 WQFT of Bi-adjoint scalar, Yang-Mills and Dilaton Gravity

In addition to the worldline coupled to Einstein gravity, we also wish to apply the WQFT formalism to massive point particles coupled to the bi-adjoint scalar field theory (BS), Yang-Mills theory (YM), and dilaton-gravity (DG). The actions can be uniformly written in a compact form,

$$S_{\mathrm{WQFT}} = S_{\mathrm{BS/YM/DG}} + \sum_i S^{(i)}_{\mathrm{cc/pc/pm}}, \tag{3.16}$$

where $S_{\mathrm{BS/YM/DG}}$ is the respective field theory action and $S^{(i)}_{\mathrm{cc/pc/pm}}$ the respective i'th particle worldline action.

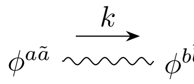
Let us first look at the field theory actions. The Yang-Mills action is the same as given in (1.1) with Feynman gauge (1.22). The gluon propagator is given in (1.23) and the three- and four-gluon vertices are (1.24) and (1.25), respectively. These are all the Feynman rules of YM field theory that we need in this chapter.

The action of bi-adjoint scalar theory is

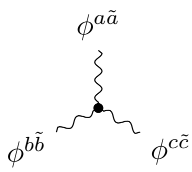
$$S_{\mathrm{BS}} = \int \mathrm{d}^4 x \left(\frac{1}{2}\left(\partial_\mu \phi^{a\tilde{a}}\right)^2 - \frac{y}{3} f^{abc} \tilde{f}^{\tilde{a}\tilde{b}\tilde{c}} \phi_{a\tilde{a}}\phi_{b\tilde{b}}\phi_{c\tilde{c}}\right) \tag{3.17}$$

where $\phi_{a\tilde{a}}(x)$ is the bi-adjoint scalar field carrying two distinct color indices a and $\tilde{a}$ related to the color and dual-color gauge groups, respectively, and y is the coupling constant. f^{abc}

and $\tilde{f}^{\tilde{a}\tilde{b}\tilde{c}}$ are corresponding structure constants. We can easily deduce the Feynman rules in the bi-adjoint scalar theory,



$$= \frac{i}{k^2} \tag{3.18}$$



$$= -2iyf^{abc}\tilde{f}^{\tilde{a}\tilde{b}\tilde{c}}. \tag{3.19}$$

As far as the double copy is concerned, one would expect to get the two-form-dilaton gravity corresponding to YM theory. However, in the classical worldline formalism, due to the anti-symmetry of the B-field, it decouples from the massive particles (at least to next-to-leading order and minimal coupling). Therefore, in WQFT, we only need to consider dilaton gravity. The action is given in (2.32) without the last term in the bracket involving the B-field. Limited to $D = 4$, it reads,

$$S_{\mathrm{dg}} = \int \mathrm{d}^4x \mathcal{L}_{\mathrm{dg}} = \int \mathrm{d}^4x \sqrt{-g}\Big[-\frac{2}{\kappa^2}R + g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\Big]. \tag{3.20}$$

After doing the weak field expansion, we follow the procedure in [90], which extensively simplifies our calculation. Accordingly, we adopt a special gauge-fixing condition other than the de Donder gauge and perform a field redefinition of $\phi, h_{\mu\nu}$. This allows decoupling the dilaton to the worldline up to subleading order in κ, as well as to simplify the three-graviton self-interaction as the square of the three-gluon one. The gauge-fixing term is given as

$$S_{\mathrm{gf}} = \frac{1}{\kappa^2}\int \mathrm{d}^4x\sqrt{-g}g_{\mu\nu}f^\mu f^\nu, \tag{3.21}$$

$$\begin{aligned} f^\mu =& \Gamma^\mu{}_{\nu\sigma}g^{\nu\sigma} + \frac{\kappa^2}{2}\Big[-\frac{1}{4}\left(\partial_\kappa h^{\kappa\lambda}\right)h^\mu{}_\lambda - \frac{1}{4}\left(\partial^\mu h^{\kappa\lambda}\right)h_{\kappa\lambda} + \left(\partial^\kappa h^{\mu\lambda}\right)h_{\kappa\lambda} \\ &+ \frac{3}{16}\left(\partial^\mu h^\kappa\right)h^\lambda{}_\lambda - \frac{3}{8}\left(\partial^\kappa h^\mu{}_\kappa\right)h^\lambda{}_\lambda - \frac{3}{8}\left(\partial^\lambda h^\kappa{}_\kappa\right)h^\mu{}_\lambda\Big]. \end{aligned} \tag{3.22}$$

The field redefinition is given as

$$\begin{aligned} h_{\mu\nu} \to& h_{\mu\nu} - \eta_{\mu\nu}\left(\frac{1}{2}h^\mu{}_\mu + 2\phi\right) + \kappa\left(-\frac{1}{2}h_{\mu\nu}h^\rho{}_\rho + \frac{1}{8}\eta_{\mu\nu}h^\rho{}_\rho h^\sigma{}_\sigma\right. \\ &\left.+\frac{1}{2}h_{\mu\rho}h^\rho{}_\nu - 2\phi h_{\mu\nu} + 2\phi^2\eta_{\mu\nu} + \phi h_{\mu\nu}h^\rho{}_\rho\right), \end{aligned} \tag{3.23a}$$

$$\phi \to \phi + \frac{1}{4}h^\mu{}_\mu. \tag{3.23b}$$

Additionally, we also add a total derivative term to simplify the action

$$\begin{aligned} 0 = S_{\mathrm{TD}} = \int \mathrm{d}^4x\partial_\mu\Big[&h^\nu{}_\kappa\partial_\nu h^{\mu\kappa} - h^{\mu\nu}\partial_\kappa h^\kappa{}_\nu \\ &+ \frac{\kappa}{4}\Big(h^{\mu\nu}h_{\sigma\nu}\partial_\nu h^{\sigma\nu} - h_{\lambda\kappa}h^{\kappa\nu}\partial_\nu h^{\mu\lambda} - h^{\mu\nu}h^{\rho\lambda}\partial_\lambda h_{\nu\rho} + h^\mu{}_\nu h^{\nu\lambda}\partial_\sigma h^\sigma_\lambda\Big)\Big] \end{aligned} \tag{3.24}$$

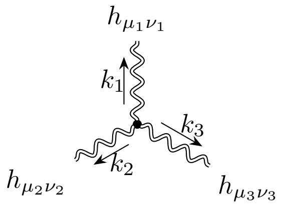
This procedure leads to a simple expression for the field action,

$$\begin{aligned} S_{\mathrm{dg}} + S_{\mathrm{gf}} + S_{\mathrm{TD}} &= \int \mathrm{d}^4x \bigg[\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \frac{\kappa}{4}\Big(h_{\mu\nu}\partial^\mu\partial^\nu h_{\rho\sigma}h^{\rho\sigma} + 2\, h_{\mu\nu}\partial^\sigma h^\mu_\rho \partial^\nu h^\rho_\sigma \\ &\qquad - h_{\mu\nu}\partial^\sigma h^\mu_\rho \partial^\rho h^\nu_\sigma - h_{\rho\sigma}\partial^\rho h_{\mu\nu}\partial^\sigma h^{\mu\nu} - \partial_\rho\partial_\sigma h_{\mu\nu}h^{\rho\mu}h^{\sigma\nu}\Big)\bigg] + \mathcal{O}\left(\kappa^2, \phi\right) \\ &= \int \mathrm{d}^4x \left[\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \frac{\kappa}{4\cdot 3!}\mathcal{V}^{\mu\alpha\gamma}_{123}\mathcal{V}^{\nu\beta\delta}_{123} h_{1\mu\nu}h_{2\alpha\beta}h_{3\gamma\delta}\right] + \mathcal{O}\big(\kappa^2, \phi\big), \end{aligned} \tag{3.25}$$

where $\mathcal{V}^{\nu\beta\delta}_{123} = V^{\nu\beta\delta}_{123}\Big|_{k_i\to\partial_i}$ is the position space version of $V^{\nu\beta\delta}_{123}$ defined in (1.26) for the three-gluon vertex, with the labels $1, 2, 3$ indicating on which $h_{\mu\nu}$ the partial derivatives should be applied. This yields the graviton propagator

$$h_{\mu\nu} \overset{k}{\rightsquigarrow} h_{\rho\sigma} = \frac{i}{k^2}\eta^{\mu(\rho}\eta^{\sigma)\nu} \tag{3.26}$$

and the three-graviton vertex



$$= \frac{-i\kappa}{4} V^{\alpha_1\alpha_2\alpha_3}_{123} V^{\beta_1\beta_2\beta_3}_{123} \prod_{i=1}^{3} \eta_{\alpha_i(\mu_i}\eta_{\nu_i)\beta_i}. \tag{3.27}$$

Compared to the three-gluon vertex (1.24), we can already see the structure of the double copy.

We will now proceed to examine the worldline actions $S^{(i)}_{\mathrm{cc/pc/pm}}$ in (3.16). The action of a massive point charge coupled to a non-abelian gauge field A^a_μ is [91, 92]

$$S_{\mathrm{pc}} = -\int \mathrm{d}\tau \left(\frac{1}{2}\left(\frac{\dot{x}^2}{e} + m^2 e\right) - i\Psi^{\dagger\alpha}\dot{\Psi}_\alpha + g\dot{x}^\mu A^a_\mu C^a\right), \tag{3.28}$$

where $e(\tau)$ is the einbein similar to the one introduced for gravity (1.79), and the dot over a symbol denotes a derivative with respect to τ. The "color wave function" $\Psi_\alpha(\tau)$ is an auxiliary field carrying the color degrees of freedom of the particle, the $\alpha, \beta, \ldots = 1, \ldots, d_R$ are indices of the d_R-dimensional representation of the gauge group, and

$$C^a(\tau) = \Psi^{\dagger\alpha}(T^a)_\alpha{}^\beta\Psi_\beta \tag{3.29}$$

is the associated color charge that determines the coupling to the gauge field $A^a_\mu(x)$. We shall take the generators $(T^a)_\alpha{}^\beta$ to be acting on the fundamental of $SU(N)$ such that $d_R = N$ and the adjoint indices $a, b, \ldots = 1, \ldots N^2 - 1$.[1] This action is invariant under the reparametrization of τ. The kinetic term can be transformed into the more familiar form $-m\int \mathrm{d}\tau\sqrt{\dot{x}^2}$ by solving the algebraic equations of motion for the einbein $e(\tau)$ and reinserting

[1] In fact, specifying the gauge group and the representation for the matter are not necessary. In principle, we can pick any group and representation. We pick the fundamental one just for convenience.

the solution into the action just as for Einstein gravity (1.80). However, as usual, it is better for us to fix

$$e(\tau) = \frac{1}{m} \qquad \Rightarrow \qquad \dot{x}^2 = 1 \tag{3.30}$$

and τ is then the proper time.

Similarly, the action of a worldline minimally coupled to dilaton-gravity reads

$$S_{\text{pm}} = -\frac{1}{2}\int \mathrm{d}\tau \left(\frac{e^{2\kappa\varphi} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{e} + m^2 e \right), \tag{3.31}$$

which differs from the pure-gravity-worldline action (1.78) only by the exponentiated dilaton coupling $e^{2\kappa\varphi}$. Again, upon integrating out $e(\tau)$ we arrive at the more common form of the action $-m\int \mathrm{d}\tau e^{\kappa\varphi}\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$. We fix the worldline gauge by choosing

$$e(\tau) = \frac{1}{m} \qquad \Rightarrow \qquad e^{2\kappa\varphi} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 1. \tag{3.32}$$

Thanks to the field redefinition (3.23), the worldline is decoupled from φ up to quadratic order, which in terms of the redefined fields then reads

$$S_{\text{pm}} = -\frac{m}{2}\int \mathrm{d}\tau \Big(\dot{x}^2 + \kappa h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{\kappa^2}{2} h_{\mu\rho} h_\nu{}^\rho \dot{x}^\mu \dot{x}^\nu \Big) + \mathcal{O}(\kappa^3). \tag{3.33}$$

Finally, let us introduce the massive point particle coupling to a bi-adjoint scalar field theory. A point particle interacting with a bi-adjoint scalar field (WBS) is described by [93,94]

$$S_{\text{cc}} = -\int \mathrm{d}\tau \Big(\frac{1}{2}\left(\frac{\dot{x}^2}{e} + m^2 e \right) - i\Psi^\dagger \dot{\Psi} - i\tilde{\Psi}^\dagger \dot{\tilde{\Psi}} - e\, y\, \phi_{a\tilde{a}} C^a \tilde{C}^{\tilde{a}} \Big), \tag{3.34}$$

where $\Psi_\alpha(\tau)$ and $\tilde{\Psi}_{\tilde{\alpha}}(\tau)$ are the color and dual color wave functions, and we have omitted the (dual-)color indies for brevity. The corresponding charges are defined in a similar fashion as (3.29) in S_{pc}

$$C^a = \Psi^\dagger T^a \Psi, \qquad \tilde{C}^{\tilde{a}} = \tilde{\Psi}^\dagger \tilde{T}^{\tilde{a}} \tilde{\Psi}. \tag{3.35}$$

In this case, we set the worldline parametrization

$$e(\tau) = \frac{1}{m} \qquad \Rightarrow \qquad \dot{x}^2 + \frac{2y}{m^2}\phi_{a\tilde{a}} C^a \tilde{C}^{\tilde{a}} = 1. \tag{3.36}$$

Following the spirit of the pure gravity case, we will expand the worldline coordinate $x^\mu(\tau)$ along a background straight line defined in the past infinity as (3.2). We are thus obligated to use retarded propagators for physical observables. However, in the next section, we will be focusing on the calculation of the eikonal, which does not have a natural direction of time, and the type of propagator is hence uncertain. Fortunately, as in this work our main concern for the double copy construction is the integrand, so the $i\varepsilon$ description of the propagators is of no direct concern. Therefore we may this obstruction.

In addition to the straight-line expansion, we also decompose the color wave function in the background,

$$\Psi(\tau) = \psi + \mathit{\Psi}(\tau), \tag{3.37}$$

where the lowercase $\psi = \Psi(-\infty) = \text{const}$ is the initial condition, and calligraphic uppercase $\mathit{\Psi}(\tau)$ is the fluctuation that will be quantized. Consequently, the color charge is

$$C^a = c^a + \psi^\dagger T^a \mathit{\Psi} + \mathit{\Psi}^\dagger T^a \psi + \mathit{\Psi}^\dagger T^a \mathit{\Psi}, \tag{3.38}$$

where we have defined the background color charge

$$c^a = \psi^\dagger T^a \psi. \tag{3.39}$$

A similar decomposition applies to the dual-color wave function $\tilde{\Psi}(\tau)$, and all respective dual quantities are denoted with a tilde. Note that the background $x(\tau) = b^\mu + v^\mu \tau$ and $\Psi(\tau) = \psi$ solves the equations of motion in the field free scenario(s) $\phi_{a\tilde{a}} = A^a_\mu = h_{\mu\nu} = \varphi = 0$.

In WQFT, as illustrated before, we integrate both the field in the bulk and fluctuations of the worldlines. Specific to the cases we consider here, we will integrate out the BS/YM/DG fields $\phi_{a\tilde{a}}; A^a_\mu; h_{\mu\nu}, \varphi$ as well as all fluctuations of worldline degrees of freedom $z(\tau), \mathit{\Psi}(\tau), \tilde{\mathit{\Psi}}(\tau)$ in the path integral, so the results only depend on the background fields b, v, ψ. In the path integral, similar to (3.4), the expectation value of an operator $\mathcal{O}$ is expressed as

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}_{\text{WQFT}}} \int D[\Phi] \prod_i D[z_i] \left(D[\mathit{\Psi}_i, \tilde{\mathit{\Psi}}_i] \right) \mathcal{O}\, e^{iS_{\text{WQFT}}}, \tag{3.40}$$

where $\Phi \in \{\phi_{a\tilde{a}}, A^a_\mu, h_{\mu\nu}, \varphi\}$ denotes the bosonic fields in the respective theories. The partition function follows,

$$\mathcal{Z}_{\text{WQFT}} = \int D[\Phi] \prod_i D[z_i] \left(D[\mathit{\Psi}_i, \tilde{\mathit{\Psi}}_i] \right) e^{iS_{\text{WQFT}}}. \tag{3.41}$$

Moving to momentum space, we express the color wave function fluctuation in the same way with (3.12),

$$\begin{aligned} \mathit{\Psi}(\tau) &= \int_\omega e^{-i\omega\tau} \mathit{\Psi}(\omega), \\ \mathit{\Psi}^\dagger(\tau) &= \int_\omega e^{-i\omega\tau} \mathit{\Psi}^\dagger(-\omega). \end{aligned} \tag{3.42}$$

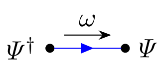
The same also applies to the dual-color wave function $\tilde{\mathit{\Psi}}$. When evaluated on the worldline, the generic field $\Phi(x(\tau))$ may be expanded in the same way as the pure graviton (3.14),

$$\begin{aligned} \Phi(x(\tau)) &= \int_k e^{ik\cdot(b+v\tau+z(\tau))} \Phi(-k) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_k e^{ik\cdot(b+v\tau)} (k\cdot z(\tau))^n \Phi(-k) \\ &= \int_k e^{ik\cdot b} \Phi(-k) \left(e^{ik\cdot v\tau} + i \int_\omega e^{i(k\cdot v+\omega)\tau} k\cdot z(-\omega) \right) + \mathcal{O}(z^2). \end{aligned} \tag{3.43}$$

We take the expansion only to linear order in z^μ since this is the highest term we need in this chapter. A complete expression of $h_{\mu\nu}$ to all orders in z may be found in [84].

3.1.2 WQFT Feynman Rules

Next, we extract the Feynman rules from the worldline actions. The worldline propagators are the same in all three theories. In addition to the $z(\tau)$ propagators, we also have the color wave function fluctuation $\mathit{\Psi}(\tau)$,

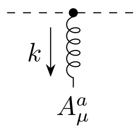


$$= \frac{i}{\omega}. \tag{3.44}$$

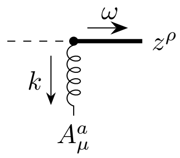
The propagator of the dual field $\tilde{\Psi}$ is identical to the one for Ψ. Let us now begin with the analysis of the Yang-Mills coupled WQFT. With (3.12), (3.42) and (3.43) we can expand the interaction term of S_{pc} from eq. (3.28) as

$$\begin{aligned} S_{\mathrm{pc}}^{\mathrm{int}} :=& -g\int \mathrm{d}\tau\, \dot{x}^\mu(\tau)\cdot A^a(x(\tau))\, C^a(\tau) \\ =& -g\int_k e^{ik\cdot b} v\cdot A^a(-k)\hat{\delta}(k\cdot v)c^a - g\int_{k,\omega} e^{ik\cdot b}A^a_\mu(-k)\hat{\delta}(k\cdot v+\omega) \\ &\times\Big[i\big(\omega z^\mu(-\omega)+v^\mu k\cdot z(-\omega)\big)c^a + v^\mu(\psi^\dagger T^a\Psi(-\omega)+\Psi^\dagger(\omega)T^a\psi)\Big]+\mathcal{O}\left((z,\Psi)^2\right), \end{aligned} \tag{3.45}$$

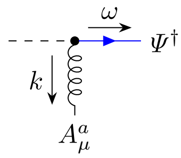
where we keep the interaction to linear order in worldline fluctuations. The Feynman rules of the worldline-gluon vertices can be directly read off from (3.45). Below we represent the background worldline configurations (b^μ, v^μ, c^a) as dashed lines:



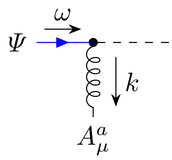
$$= -ige^{ik\cdot b}\hat{\delta}(k\cdot v)v^\mu c^a \tag{3.46}$$



$$= ge^{ik\cdot b}\hat{\delta}(k\cdot v+\omega)(\omega\,\eta^{\mu\rho}+v^\mu k^\rho)c^a \tag{3.47}$$



$$= -\ ige^{ik\cdot b}\hat{\delta}(k\cdot v+\omega)v^\mu(T^a\psi) \tag{3.48}$$



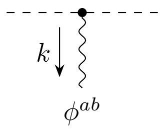
$$= -\ ige^{ik\cdot b}\hat{\delta}(k\cdot v-\omega)v^\mu(\psi^\dagger T^a), \tag{3.49}$$

where we have used the shorthand notation $\hat{\delta}$ defined in (3.13).

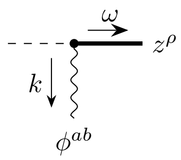
Turning to the bi-adjoint scalar coupled WQFT, we can expand the worldline-scalar coupling of (3.34) in the same way,

$$\begin{aligned} S_{\mathrm{cc}}^{\mathrm{int}} :=& \frac{y}{m}\int \mathrm{d}\tau \phi^{a\tilde{a}}(x(\tau))C^a(\tau)C^{\tilde{a}}(\tau) \\ =& \frac{y}{m}\int_k e^{ik\cdot b}\phi^{a\tilde{a}}(-k)\hat{\delta}(k\cdot v)c^a c^{\tilde{a}} + \frac{y}{m}\int_{k,\omega} e^{ik\cdot b}\phi^{a\tilde{a}}(-k)\hat{\delta}(k\cdot v+\omega) \\ &\times\Big[ik\cdot z(-\omega)c^a c^{\tilde{a}} + \Big(\psi^\dagger T^a\Psi(-\omega)+\Psi^\dagger(\omega)T^a\psi\Big)c^{\tilde{a}} \\ &\quad + c^a\Big(\tilde{\psi}^\dagger\tilde{T}^{\tilde{a}}\tilde{\Psi}(-\omega)+\tilde{\Psi}^\dagger(\omega)\tilde{T}^{\tilde{a}}\tilde{\psi}\Big)\Big]+\mathcal{O}\left((z,\Psi)^2\right). \end{aligned} \tag{3.50}$$

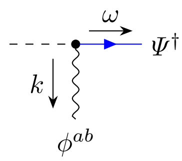
Again, we keep only the terms that we need in this work. From the interaction (3.50) we extract the Feynman rules



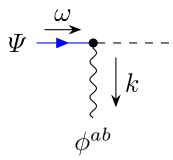
$$= \frac{iy}{m} e^{ik\cdot b} \hat{\delta}(k\cdot v) c^a c^{\tilde{a}} \tag{3.51}$$



$$= -\frac{y}{m} e^{ik\cdot b} \hat{\delta}(k\cdot v+\omega) k^\rho c^a c^{\tilde{a}} \tag{3.52}$$



$$= \frac{iy}{m} e^{ik\cdot b} \hat{\delta}(k\cdot v+\omega) (T^a \psi) c^{\tilde{a}} \tag{3.53}$$



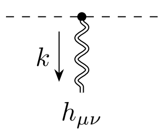
$$= \frac{iy}{m} e^{ik\cdot b} \hat{\delta}(k\cdot v-\omega) (\psi^\dagger T^a) c^{\tilde{a}}. \tag{3.54}$$

For vertices that involves the dual wave function, we simply use (3.53) or (3.54) and change Ψ to $\tilde{\Psi}$.

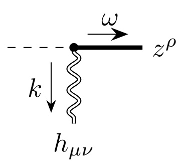
In the dilaton-gravity coupled WQFT, the interaction term is remarkably simplified by the field redefinitions (3.23) of $\{\varphi, h_{\mu\nu}\}$. In the end the linear order in $h_{\mu\nu}$ in (3.33) is no different than the interaction term of a point mass in pure gravity (3.15). From (3.33), we here explicitly provide the leading terms in $z(\omega)$,

$$\begin{aligned} S^{\text{int}}_{\text{pm}} = &-\frac{m\kappa}{2}\int_k e^{ik\cdot b}\hat{\delta}(k\cdot v) h_{\mu\nu}(-k) v^\mu v^\nu \\ &- i\frac{m\kappa}{2}\int_{k,\omega} e^{ik\cdot b}\hat{\delta}(k\cdot v+\omega) h_{\mu\nu}(-k) z^\rho(-\omega)\left(2\omega v^{(\mu}\delta^{\nu)}_\rho + v^\mu v^\nu k_\rho\right) \\ &- \frac{m\kappa^2}{4}\int_{k_1,k_2} e^{(k_1+k_2)\cdot b}\hat{\delta}((k_1+k_2)\cdot v) h_{\mu\rho}(-k_1) h_\nu{}^\rho(-k_2) v^\mu v^\nu + \mathcal{O}(k^3, z^2). \end{aligned} \tag{3.55}$$

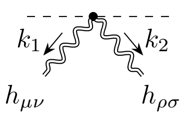
We obtain the Feynman rules,



$$= -i\frac{m\kappa}{2} e^{ik\cdot b}\hat{\delta}(k\cdot v) v^\mu v^\nu \tag{3.56}$$



$$= \frac{m\kappa}{2} e^{ik\cdot b}\hat{\delta}(k\cdot v+\omega)\left(2\omega v^{(\mu}\delta^{\nu)}_\rho + v^\mu v^\nu k_\rho\right) \tag{3.57}$$



$$= -\frac{m\kappa^2}{2}\int_{k_1,k_2} e^{i(k_1+k_2)\cdot b}\hat{\delta}\left((k_1+k_2)\cdot v\right) v^{(\mu}\eta^{\nu)(\rho}v^{\sigma)}. \tag{3.58}$$

3.2 Eikonal Phase and the Double Copy

One of the main challenges of constructing the double copy in the classical limit of quantum field theories is that the locality structure is concealed. This is rooted in the classical limit of the massive scalar propagator [84], which contains both double and single propagators as we can see in WQFT from (3.11) and (3.44). Following [93] we tackle this difficulty by using the bi-adjoint scalar theory to identify the correct locality structure, i.e. disentangle the kinematical numerators from the propagator terms.

Another important strategy to establish the classical double copy is to consider more than two worldlines even if we are ultimately interested only in two-body interactions. This is to avoid the situation where some color factors in the two-body situation are vanishing but the corresponding numerators do not, which under the double copy map may yield non-zero contributions. This may be evaded if we use as many worldlines as worldline-field interactions occur. Specifically, we will consider an $(n+2)$-body system at N^{n}LO. To retrieve the binary system from this, we need to sum all possible ways of fusing the $(n+2)$ worldlines into 2 worldlines. In summary, our double copy relation of the eikonal phase at $\mathrm{N}^{(\mathrm{n}-1)}$LO reads

$$\chi_n^{\mathrm{BS}} = -\, y^{2n} \int \mathrm{d}\mu_{1,2,\ldots,(n+1)} \sum_{i,j} C_i K_{ij} \tilde{C}_j, \tag{3.59a}$$

$$\chi_n^{\mathrm{YM}} = -\, (ig)^{2n} \int \mathrm{d}\mu_{1,2,\ldots,(n+1)} \sum_{i,j} C_i K_{ij} N_j, \tag{3.59b}$$

$$\chi_n^{\mathrm{DG}} = -\left(\frac{\kappa}{2}\right)^{2n} \int \mathrm{d}\mu_{1,2,\ldots,(n+1)} \sum_{i,j} N_i K_{ij} N_j, \tag{3.59c}$$

where $C_i, \tilde{C}_j$ denotes the color and dual color factors, N_j are the numerators, and K_{ij} are the so-called double copy kernels that encode the locality structure. The sums extend over the dimensionalities of the numerators and the color factors. For further convenience, we have also defined the integral measure

$$\mathrm{d}\mu_{1,2,\ldots,n} = \prod_{i=1}^{n} \left(\frac{\mathrm{d}^4 k_i}{(2\pi)^4} e^{ik_i\cdot b_i} \delta\!\!\!^{-}(k_i\cdot p_i)\right) \delta\!\!\!^{-(4)}\left(\sum_{i=1}^{n} k_i^{\mu}\right), \tag{3.60}$$

where k_i is the total outgoing momentum of bosonic fields $\Phi(x)$ attached to a worldline. Note that we have defined the momentum of the massive particle as

$$p_i^{\mu} := m_i v_i^{\mu}, \quad \text{so that} \quad \delta\!\!\!^{-}(k_i\cdot p_i) = \frac{\delta\!\!\!^{-}(k_i\cdot v_i)}{m_i}. \tag{3.61}$$

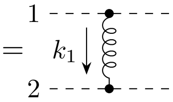
Hereafter we will always express the numerator N_j in terms of the momentum p_i^{μ} which is necessary in order to balance the mass dimension under the double copy. The kinematic numerators N_i are arranged to satisfy the same algebraic equations as the color factors C_i,

$$C_i + C_j + C_k = 0 \quad \Rightarrow \quad N_i + N_j + N_k = 0. \tag{3.62}$$

It is worth mentioning that we have the color-kinematic duality already at quartic order in the coupling constant.

3.2.1 Eikonal at Leading Order (LO)

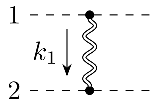
Let us first consider the double copy of the eikonal at leading order. The locality structure at leading order is trivial, so we do not need to employ the bi-adjoint scalar theory in order to double copy YM color charged particles to DG ones. In Yang-Mills coupled WQFT (WYM) the eikonal phase at this order involves only one diagram. Using the Feynman rules (3.46) and the gluon propagator (1.23), we have

$$i\chi_1^{\text{YM}} =$$  $$= ig^2 \int \mathrm{d}\mu_{1,2} \frac{(p_1 \cdot p_2)(c_1 \cdot c_2)}{k_1^2}, \tag{3.63}$$

where we have massaged the formula to fit the form as (3.59b). It is straightforward to identify the color factor, the numerator and the double copy kernel

$$C = (c_1 \cdot c_2), \quad N = (p_1 \cdot p_2), \quad K = \frac{1}{k_1^2}. \tag{3.64}$$

In worldline coupled dilaton-gravity (WDG), thanks to the decoupling of φ from the worldline, we also have only one diagram mediated by $h_{\mu\nu}$. With (3.56) and the graviton propagator (1.40), we obtain

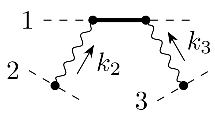
$$i\chi_1^{\text{DG}} =$$  $$= \frac{-i\kappa^2}{4} \int \mathrm{d}\mu_{1,2} \frac{(p_1 \cdot p_2)^2}{k_1^2}. \tag{3.65}$$

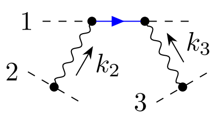
Hence at the leading order, the eikonal of Yang-Mills and dilaton gravity obviously possess a double copy relation (3.59).

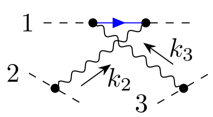
3.2.2 Eikonal at Next-to-Leading Order (NLO)

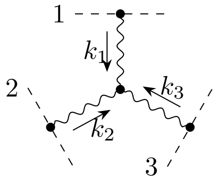
As explained before, at next-to-leading order, to avoid the vanishing of some contributions in worldline coupled bi-adjoint scalar theory (WBS) and Yang-Mills coupled WQFT theory, we will consider three worldlines. At this order the locality structure is non-trivial. As we will see, the double copy kernel is off-diagonal. Therefore, we will first consider the bi-adjoint scalar theory to identify the kernel.

The Feynman diagrams in WBS can be calculated using the Feynman rules (3.51)-(3.54) and the three-point vertex of $\phi_{a\tilde{a}}$ (3.19),

 $$= -iy^4 \int \frac{\mathrm{d}\mu_{1,2,3}}{k_2^2 k_3^2} \frac{k_2 \cdot k_3}{(k_2 \cdot p_1)^2} (c_1 \cdot c_2)(c_1 \cdot c_3)(\tilde{c}_1 \cdot \tilde{c}_2)(\tilde{c}_1 \cdot \tilde{c}_2) \tag{3.66}$$

 $$= -iy^4 \int \frac{\mathrm{d}\mu_{1,2,3}}{k_2^2 k_3^2} \frac{1}{k_2 \cdot p_1} \big((c_1^{ba} c_2^a c_3^b)(\tilde{c}_1 \cdot \tilde{c}_2)(\tilde{c}_1 \cdot \tilde{c}_3) + (c_1 \cdot c_2)(c_1 \cdot c_3)(\tilde{c}_1^{\tilde{b}\tilde{a}} \tilde{c}_2^{\tilde{a}} \tilde{c}_3^{\tilde{b}})\big) \tag{3.67}$$

 $$= -iy^4 \int \frac{\mathrm{d}\mu_{1,2,3}}{k_2^2 k_3^2} \frac{-1}{k_2 \cdot p_1} \big((c_1^{ab} c_2^a c_3^b)(\tilde{c}_1 \cdot \tilde{c}_2)(\tilde{c}_1 \cdot \tilde{c}_3) + (c_1 \cdot c_2)(c_1 \cdot c_3)(\tilde{c}_1^{\tilde{a}\tilde{b}} \tilde{c}_2^{\tilde{a}} \tilde{c}_3^{\tilde{b}})\big) \tag{3.68}$$



$$= -iy^4 \int \frac{\mathrm{d}\mu_{1,2,3}}{k_1^2 k_2^2 k_3^2} \, 2 f^{abc} c_1^a c_2^b c_3^c \tilde{f}^{\tilde{a}\tilde{b}\tilde{c}} \tilde{c}_1^{\tilde{a}} \tilde{c}_2^{\tilde{b}} \tilde{c}_3^{\tilde{c}} \tag{3.69}$$

where for compactness we have defined

$$c^{ab} := \left(\psi^\dagger T^a T^b \psi\right) , \quad \tilde{c}^{\tilde{a}\tilde{b}} := \left(\tilde{\psi}^\dagger \tilde{T}^{\tilde{a}} \tilde{T}^{\tilde{b}} \tilde{\psi}\right) . \tag{3.70}$$

Note that in (3.67) and (3.68), the propagator with an arrow denotes either the color or dual color wave function, and we have added up their contributions. We stress that the factors c^{ab} are absent in the equation of motion, so they will not explicitly appear in the classical solutions [95]. In fact, summing up (3.67) and (3.68) we can remove c^{ab} by

$$c^{ab} - c^{ba} = f^{abc} c^c \,, \tag{3.71}$$

and similarly for the dual-color sector. However, these factors turn out to be critical for the double copy: because of them we find classical numerators that satisfy color-kinematics duality at this order. From (3.66) - (3.68) we can identify 3 (dual-)color factors,

$$C_i^{(123)} = \left\{ (c_1 \cdot c_2)(c_1 \cdot c_3) \,, \, (c_1^{ab} c_2^a c_3^b) \,, \, (c_1^{ba} c_2^a c_3^b) \right\} \tag{3.72}$$

$$\tilde{C}_i^{(123)} = \left\{ (\tilde{c}_1 \cdot \tilde{c}_2)(\tilde{c}_1 \cdot \tilde{c}_3) \,, \, (\tilde{c}_1^{\tilde{a}\tilde{b}} \tilde{c}_2^{\tilde{a}} \tilde{c}_3^{\tilde{b}}) \,, \, (\tilde{c}_1^{\tilde{b}\tilde{a}} \tilde{c}_2^{\tilde{a}} \tilde{c}_3^{\tilde{b}}) \right\} . \tag{3.73}$$

Note that here we only consider diagrams with worldline propagators of particle 1. There are also contributions involving propagators of 2 and 3, which can be gained simply by relabeling (123) in (3.66)-(3.68) and give us another 6 color factors. Together with the single (dual)-color factor emerging from (3.69)

$$C_i^{(0)} = f^{abc} c_1^a c_2^b c_3^c, \qquad \tilde{C}_i^{(0)} = \tilde{f}^{\tilde{a}\tilde{b}\tilde{c}} \tilde{c}_1^{\tilde{a}} \tilde{c}_2^{\tilde{b}} \tilde{c}_3^{\tilde{c}}, \tag{3.74}$$

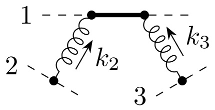
we see that the double copy kernel K_{ij} is 10-dimensional. Fortunately, K_{ij} is block-diagonal. The block that corresponds to the three-dimensional space (3.72) is

$$K_{ij}^{(123)} = \frac{1}{k_2^2 k_3^2} \begin{pmatrix} \frac{k_2 \cdot k_3}{(k_2 \cdot p_1)^2} & \frac{-1}{k_2 \cdot p_1} & \frac{1}{k_2 \cdot p_1} \\ \frac{-1}{k_2 \cdot p_1} & 0 & 0 \\ \frac{1}{k_2 \cdot p_1} & 0 & 0 \end{pmatrix} \tag{3.75}$$

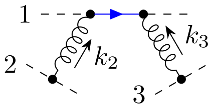
and analogously for the color-dual (3.73). By permutations of (123) we may obtain other blocks. The last block coupling to the structure constant is extracted from (3.69) and is 1-dimensional, thus we have

$$K_{ij}^{(0)} = \frac{2}{k_1^2 k_2^2 k_3^2} . \tag{3.76}$$

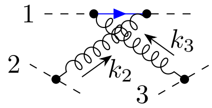
We now proceed to consider the Yang-Mills coupled WQFT (WYM) theory. The Feynman diagrams are very similar to those of WBS theory. With the WYM Feynman rules (3.46) - (3.49), we may compute the contributions to the eikonal phase



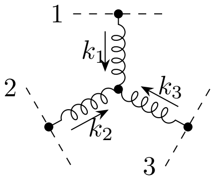
$$= -ig^4 \int \frac{\mathrm{d}\mu_{1,2,3}}{k_2^2 k_3^2} (c_1 \cdot c_2)(c_1 \cdot c_3) \left(\frac{k_2 \cdot k_3}{(k_2 \cdot p_1)^2} n_0 + \frac{1}{k_2 \cdot p_1} n_1 \right) \tag{3.77}$$



$$= -ig^4 \int \frac{\mathrm{d}\mu_{1,2,3}}{k_2^2 k_3^2} \frac{(c_1^{ba} c_2^a c_3^b)}{k_2 \cdot p_1} n_0 \tag{3.78}$$



$$= -ig^4 \int \frac{\mathrm{d}\mu_{1,2,3}}{k_2^2 k_3^2} \frac{-(c_1^{ab} c_2^a c_3^b)}{k_2 \cdot p_1} n_0 \tag{3.79}$$



$$= -ig^4 \int \mathrm{d}\mu_{1,2,3} \frac{2 f^{abc} c_1^a c_2^b c_3^c}{k_1^2 k_2^2 k_3^2} (-n_1) \tag{3.80}$$

where we have defined

$$n_0 = p_1 \cdot p_2 \, p_1 \cdot p_3 \tag{3.81}$$
$$n_1 = k_2 \cdot p_3 \, p_1 \cdot p_2 - k_3 \cdot p_2 \, p_1 \cdot p_3 - k_2 \cdot p_1 \, p_2 \cdot p_3 . \tag{3.82}$$

Based on the color factors identified in (3.72), (3.74) and the double copy kernel (3.75), (3.76), we are led to organize the numerators as

$$N_j^{(123)} = \left\{ n_0 \, , \, \frac{-n_1}{2} \, , \, \frac{n_1}{2} \right\} \tag{3.83}$$
$$N_j^{(0)} = - \, n_1 \, , \tag{3.84}$$

so that the WYM eikonal may be decomposed in the form of (3.59b),

$$\chi_2 = -g^4 \int \mathrm{d}\mu_{1,2,3} \sum_{i,j} \Big(C_i^{(0)} K_{ij}^{(0)} N_j^{(0)} + \big(C_i^{(123)} K_{ij}^{(123)} N_j^{(123)} + \text{cyclic} \big) \Big). \tag{3.85}$$

Fortunately, this decomposition automatically satisfies the color-kinematics duality

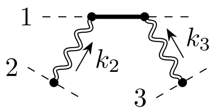
$$c_1^{ab} c_2^a c_3^b - c_1^{ba} c_2^a c_3^b = f^{abc} c_1^a c_2^b c_3^c \tag{3.86}$$
$$\frac{-n_1}{2} - \frac{n_1}{2} = - \, n_1 . \tag{3.87}$$

Note that the decomposition of $N_j^{(123)}$ is not unique due to the Jacobi relation (3.71)[2]. We note that the color-kinematic duality is satisfied automatically up to this leading and next-to-leading order if one uses Feynman gauge (this is not so in other gauges). We expect this property to break at higher orders in perturbation theory where the need of generalized gauge transformations arises. One then adds to the eikonal an arbitrary function multiplying the color factor Jacobi identity (eq. (3.86)) in order to create a color-kinematic duality respecting representation.

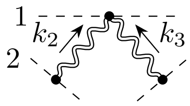
In principle, we are now prepared to execute the double copy as proposed in (3.59c) to get the eikonal phase in the worldline coupled dilaton gravity theory (WDG). In order to

[2]For example, we could also have $N_j^{(123)} = (n_0 \quad 0 \quad n_1)$. Color-kinematic duality still holds and the double copy gives the correct gravitational result. We have chosen to write $N_j^{(123)}$ in a symmetric form.

check the validity of our double copy prescription, we directly compute the eikonal in WDG theory with (3.56) - (3.58) we find for the graphs not involving bulk graviton interactions



$$= \frac{-i\kappa^4}{16}\int \frac{\mathrm{d}\mu_{1,2,3}}{k_2^2k_3^2(k_2\cdot p_1)^2}\Big((k_2\cdot k_3)(p_1\cdot p_2)^2(p_1\cdot p_3)^2 - 4(k_2\cdot p_1)^2(p_1\cdot p_2)(p_1\cdot p_3)(p_2\cdot p_3) - 2(k_3\cdot p_2)(k_2\cdot p_1)(p_1\cdot p_2)(p_1\cdot p_3)^2 + 2(k_2\cdot p_3)(k_2\cdot p_1)(p_1\cdot p_3)(p_1\cdot p_2)^2\Big) \tag{3.88}$$

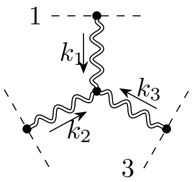


$$= \frac{-i\kappa^4}{16}\int \frac{\mathrm{d}\mu_{1,2,3}}{k_2^2k_3^2}\big(2(p_1\cdot p_2)(p_1\cdot p_3)(p_2\cdot p_3)\big). \tag{3.89}$$

Summing up the two diagrams, we can check that the result can be written as

$$\frac{-i\kappa^4}{16}\int \frac{\mathrm{d}\mu_{1,2,3}}{k_2^2k_3^2}\left(\frac{k_2\cdot k_3 n_0^2}{(k_2\cdot p_1)^2}+\frac{2n_0n_1}{k_2\cdot p_1}\right) = \frac{-i\kappa^4}{16}\int \mathrm{d}\mu_{1,2,3}\sum_{i,j} N_i^{(123)}K_{ij}^{(123)}N_j^{(123)}. \tag{3.90}$$

We have arranged the result to the form of (3.59c) with the double copy kernel $K_{ij}^{(123)}$ and the numerator $N_i^{(123)}$ defined in (3.75) and (3.83) respectively. Turning to the bulk graviton interaction graphs, thanks to the field redefinition of $\{\varphi, h_{\mu\nu}\}$, the three-graviton vertex (3.27) is directly proportional to the square of three-gluon vertex (1.24), so we can easily compute the last diagram which is manifestly a double-copy of the WYM one



$$= \frac{-i\kappa^4}{16}\int \mathrm{d}\mu_{1,2,3}\frac{2n_1^2}{k_1^2k_2^2k_3^2}. \tag{3.91}$$

From (3.90) and (3.91) we thus conclude that the double copy of the WYM eikonal coincides with the one of WDG also at the next-to-leading order ($\mathcal{O}(\kappa^4)$).

3.3 Radiation and the Double Copy

In this chapter, we are mainly considering the conservative sector of the WQFT. However, with a slight modification, we can generalize the eikonal double copy (3.59) to classical radiation. In WQFT, the Φ field radiation is computed as [85, 86]

$$-ik^2\,\langle\Phi(k)\rangle|_{k^2=0} \tag{3.92}$$

For $\Phi\in\{A_\mu^a, h_{\mu\nu}\}$, we also need to contract it with the polarizations $\{\epsilon^\mu, \epsilon^{\mu\nu}\}$ respectively. We take the gluon radiation as an example. Loosely speaking, the radiation at order $\mathcal{O}(g^{2n-1})$ can be obtained from the eikonal phase at $\mathcal{O}(g^{2n})$ by cutting off one worldline. Diagrammatically, the gluon radiation of a binary source at leading order can be gained from (3.77)-(3.80) by cutting the propagator k_3 and identifying k_3 with the momentum of the radiated gluon. The on-shell condition $k_3\cdot\epsilon = 0$ plays the same role as the $\hat{\delta}(k_3\cdot p_3)$ in the measure of the eikonal phase. This ensures that the gluon radiation can be decomposed into $C_iK_{ij}N_j$, with C_i

attained from (3.72) and (3.69) by striping off c_3, N_i from (3.83) and (3.84) by replacing p_3^μ by ϵ^μ and K_{ij} being identical to (3.75). From the same approach, we can also get the gravitational radiation and decompose it as $N_i K_{ij} N_j$. Therefore we conclude that the double copy construction works for radiation, too. We note that this is equivalent to the approach considered by Shen [93] and Goldberger and Ridgway [96] where the radiation is calculated by solving the equations of motion.

3.4 From Amplitude to Eikonal

As was mentioned at the beginning of this chapter, the expectation values in WQFT are directly linked to the classical limit of S-matrix element. Consequently, we can expect that the classical double copy of WQFT discussed is also closely related to the double copy at the level of the scattering amplitude. In this section, we will briefly introduce the procedure to get the classical limit of the scattering amplitude. Specifically, we consider the double copy of scalar QCD, which is discussed in the previous section. We claim that the classical limit of the amplitude of n distinct scalar pairs corresponds to the WYM eikonal phase at $\mathcal{O}\left(g^{2(n-1)}\right)$ and show the connection explicitly at $\mathcal{O}\left(g^4\right)$. Moreover, we will demonstrate that the double copy of the eikonal phase is the classical limit of the BCJ double copy of the scattering amplitude.

The exponentiated eikonal phase is directly related to the classical limit of scattering amplitude [97, 98],

$$(1+\Delta_q)e^{i\chi} - 1 = \sum_{n=2} \frac{1}{2^n} \int \mathrm{d}\mu_{1,2,\ldots,n} \lim_{\hbar\to 0} \mathcal{A}(n\to n) \tag{3.93}$$

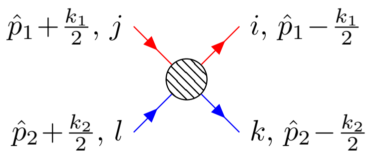
where $\mathcal{A}(n\to n)$ denotes an amplitude of n pairs of distinct massive scalars, and χ is the total eikonal phase, which scales as $\hbar^{-1}$ and receives contributions from all higher-loop amplitudes. The introduction of the "quantum remainder" Δ_q (scaling as $\hbar^{n\geq 0}$) is needed for consistency [99]. Here, we only care about tree diagrams, so we have

$$\chi_{n-1} = \frac{-i}{2^n} \int \mathrm{d}\mu_{1,2,\ldots,n} \lim_{\hbar\to 0} \mathcal{A}^{\text{tree}}(n\to n). \tag{3.94}$$

3.4.1 From SQCD amplitude and WQFT Eikonal

The correspondence at the $2\to 2$ level is relatively straightforward. The amplitude of two pairs of scalars of different flavors is (2.9). For an easier comparison with the eikonal, we rewrite it as

$\mathcal{A}^{\text{tree}}(2\to 2) =$



$$= i g^2 T^a_{ij} T^a_{lk} \frac{4\, p_1\cdot(p_2 + \frac{k_1}{2})}{k_1^2}, \tag{3.95}$$

where we have introduced $\hat{p}_i$ as the average of the in- and outgoing momentum of particle i, which is orthogonal to its momentum transfer due to the on-shellness of the external scalar,

$$0 = m_i^2 - m_i^2 = (\hat{p}_i + \frac{k_i}{2})^2 - (\hat{p}_i + \frac{k_i}{2})^2 = 2\hat{p}_i \cdot k_i \tag{3.96}$$

In the classical limit we take small momentum transfers $k_i \to \hbar k_i$, and consider the expansion in small $\hbar$ following [44]. We identify the momentum as $\hat{p}_i \to p_i$, although since $\hat{p}_i^2 \neq m_i^2$, we need to change the definition of p_i to $p_i = \hat{m}_i v_i$ with $\hat{m}_i^2 = \hat{p}_i^2$. Up to the highest order considered in this chapter, the redefinition will not change the WQFT result. Performing the classical limit of the Yang-Mills amplitude, we also need to consider the classical limit of the color factors, which was recently investigated by de la Cruz et al. [100],

$$T^a_{ij} \to c^a. \tag{3.97}$$

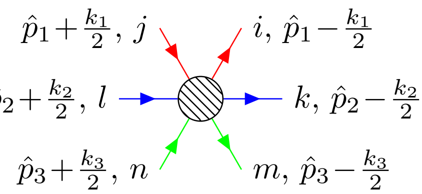
Taking the leading order in $\hbar$, we have

$$\mathcal{A}^{\text{tree}}(2 \to 2) \to ig^2(c_1 \cdot c_2)\frac{4\, p_1 \cdot p_2}{k_1^2}, \tag{3.98}$$

which coincide with the leading order eikonal (3.63) up to a prefactor. To check that the eikonal double copy agrees with that of amplitudes, we also consider the 4-scalar double copy amplitude (2.29). Following the similar procedure, we found that it agrees with (3.65).

Let us now turn to the $3 \to 3$ case. The 6−point amplitude of three distinct scalar pairs in SQCD is already given in (2.15). Nonetheless, for simpler comparison with the eikonal, we rewrite it as

$\mathcal{A}^{\text{tree}}(3 \to 3) =$



$$= \frac{8\hat{c}^{(0)}\hat{n}^{(0)}}{k_1^2 k_2^2 k_3^2} + \left[\frac{8}{k_2^2 k_3^2}\left(\frac{\hat{c}^{(123)}\hat{n}^{(123)}}{2\hat{p}_1 \cdot k_2 - k_2 \cdot k_3} + \frac{\hat{c}^{(132)}\hat{n}^{(132)}}{2\hat{p}_1 \cdot k_3 - k_3 \cdot k_2}\right) + \text{cyclic}\right]. \tag{3.99}$$

The color factors are[3]

$$\begin{aligned} \hat{c}^{(0)} &= f^{abc} T^a_{ij} T^b_{kl} T^c_{mn} \\ \hat{c}^{(123)} &= (T^b T^a)_{ij} T^a_{kl} T^b_{mn} \\ \hat{c}^{(132)} &= (T^a T^b)_{ij} T^a_{kl} T^b_{mn}. \end{aligned} \tag{3.100}$$

The corresponding numerators are

$$\hat{n}^{(0)} = -ig^4 \hat{p}_{1,\mu}\hat{p}_{2,\nu}\hat{p}_{3,\rho} V^{\mu\nu\rho}_{123} \tag{3.101}$$

$$\begin{aligned} \hat{n}^{(123)} = \frac{-ig^4}{2}\Big(& 4\hat{p}_1 \cdot \hat{p}_2\, \hat{p}_1 \cdot \hat{p}_3 + 2\hat{p}_1 \cdot \hat{p}_3\, k_1 \cdot \hat{p}_2 - 2\hat{p}_1 \cdot \hat{p}_2\, k_1 \cdot \hat{p}_3 \\ &- 2\hat{p}_1 \cdot k_2\, \hat{p}_2 \cdot \hat{p}_3 - k_1 \cdot \hat{p}_2\, k_1 \cdot \hat{p}_3 + k_2 \cdot k_3\, \hat{p}_2 \cdot \hat{p}_3\Big) \end{aligned} \tag{3.102}$$

$$\begin{aligned} \hat{n}^{(132)} = \frac{-ig^4}{2}\Big(& 4\hat{p}_1 \cdot \hat{p}_2\, \hat{p}_1 \cdot \hat{p}_3 + 2\hat{p}_1 \cdot \hat{p}_2\, k_1 \cdot \hat{p}_3 - 2\hat{p}_1 \cdot \hat{p}_3\, k_1 \cdot \hat{p}_2 \\ &- 2\hat{p}_1 \cdot k_3\, \hat{p}_2 \cdot \hat{p}_3 - k_1 \cdot \hat{p}_2\, k_1 \cdot \hat{p}_3 + k_2 \cdot k_3\, \hat{p}_2 \cdot \hat{p}_3\Big), \end{aligned} \tag{3.103}$$

[3] Note the correspondence to the color factors defined in (2.16) $\hat{c}^{(0)} \leftrightarrow c_0$, $\hat{c}^{(123)} \leftrightarrow c_{(156)}$, $\hat{c}^{(132)} \leftrightarrow c_{(134)}$.

which has been brought into a form to satisfy color-kinematic duality

$$
\begin{aligned}
\hat{c}^{(132)} - \hat{c}^{(123)} =& \hat{c}^{(0)} \\
\hat{n}^{(132)} - \hat{n}^{(123)} =& \hat{n}^{(0)}.
\end{aligned}
\tag{3.104}
$$

In (3.102) and (3.103), we have already sorted the terms in powers of k_i. The massive propagators will become

$$
\frac{1}{2\hat{p}_1\cdot k_2 - k_2\cdot k_3} \to \frac{1}{\hbar}\frac{1}{2p_1\cdot k_2} + \frac{k_2\cdot k_3}{4(p_1\cdot k_2)^2} + \mathcal{O}(\hbar). \tag{3.105}
$$

We also need to expand the color factors quadratic in the generators and linear in structure constant. Though the subleading order of $(T^aT^b)_{ij}$ is not explicitly derived yet, built on the insight of [100], we propose the classical limit of the color factors to be

$$
(T^aT^b)_{ij} \to c^ac^b + \hbar\, c^{ab} \tag{3.106}
$$

$$
f^{abc} \to \hbar f^{abc}. \tag{3.107}
$$

Note that the sub-leading term in (3.106) guarantees that the Jacobi identity holds in the classical limit.

It is now straightforward to compute the classical limit of the amplitude (3.99) and extract the eikonal using (3.94). Keeping only the leading order terms in the classical $\hbar \to 0$ limit, we have

$$
\frac{\hat{c}^{(0)}\hat{n}^{(0)}}{k_1^2k_2^2k_3^2} \to\ C_i^{(0)}K_{ij}^{(0)}N_j^{(0)} \tag{3.108}
$$

$$
\frac{1}{k_2^2k_3^2}\left(\frac{\hat{c}^{(123)}\hat{n}^{(123)}}{2\hat{p}_1\cdot k_2 - k_2\cdot k_3} + \frac{\hat{c}^{(132)}\hat{n}^{(132)}}{2\hat{p}_1\cdot k_3 - k_2\cdot k_3}\right) \to\ C_i^{(123)}K_{ij}^{(123)}N_j^{(123)}. \tag{3.109}
$$

We therefore recover the eikonal phase of SQCD from the WQFT, which directly operates at the classical level.

We can also consider the classical limit of the double copy of SQCD, which is just the 6-scalar amplitude in dilaton gravity given in (2.31). Equivalently, we can just replace the color factors with the numerators in (3.99). We can then likewise consider the classical limit of this gravitational amplitude,

$$
\frac{\hat{n}^{(0)}\hat{n}^{(0)}}{k_1^2k_2^2k_3^2} \to\ N_i^{(0)}K_{ij}^{(0)}N_j^{(0)} \tag{3.110}
$$

$$
\frac{1}{k_2^2k_3^2}\left(\frac{\hat{n}^{(123)}\hat{n}^{(123)}}{2\hat{p}_1\cdot k_2 - k_2\cdot k_3} + \frac{\hat{n}^{(132)}\hat{n}^{(132)}}{2\hat{p}_1\cdot k_3 - k_2\cdot k_3}\right) \to\ N_i^{(123)}K_{ij}^{(123)}N_j^{(123)}, \tag{3.111}
$$

which coincides with our calculation in WDG. We have therefore verified that the classical double copy of the world line quantum field theory is in full agreement with the quantum double copy of amplitudes at LO and NLO. Note that the double copy of SQCD contains self-interactions of massive scalars (2.34). However, these are short-range interactions and do not contribute to the classical theory. So we do not need to introduce additional terms in WDG, and the double copy automatically works out.

3.5 A Comment on the WQFT double copy

There is a comment we want to make regarding the double copy at the classical level. An alternative route was taken in the works [5, 90], involving the authors of this work, where a path integral based approach was taken. There, starting from the actions describing the coupling of massive, charged particles to Yang-Mills or dilaton-gravity, the force mediating fields (gluons, dilatons, and graviton) were integrated out, yielding an effective action for the point particles, thereby taking the classical $\hbar \to 0$ limit. It was shown at LO and NLO that the resulting effective action could be obtained by a suitably generalized double copy prescription [90] taking inspiration from the amplitudes approach. Concretely, the need for a trivalent graph structure was artificially introduced via delta functions on the worldline for higher valence worldline-bulk field vertices. Nevertheless, this double copy prescription was shown to break down for the effective action at the NNLO [5]. It was speculated in [5] that the reason for this breakdown lies in the attempt of double copying a gauge-variant and off-shell quantity – the effective action – which is at tension with the on-shell nature of the scattering amplitude double copy. The main difference between this previous attempt and the WQFT approach is that, in WQFT, not only do we integrate out the force mediating field in the bulk, but we also integrate out the small fluctuations of the worldline around a straight-line background. We thus arrive at results depending only on the background values, which are on-shell and gauge invariant. We are also limited to the scattering cases with a small deflection angle, which is consistent with the eikonal approximation and classical limit of scattering amplitude. Therefore, due to the parallel relation of WQFT expectation values to scattering amplitudes, we expect the double copy relation goes beyond the eikonal and radiation. In fact, all expectation values in WYM and WDG should feature the double copy relation as long as they are directly related to the quantum amplitudes.

Chapter 4

Geodesics from Classical Double Copy

This chapter is based on the published article "Geodesics From Classical Double Copy" [3], written in collaboration with Dr. Riccardo Gonzo. We will adapt the conventions for the consistency of this thesis. In particular, we change the convention of the signature of the metric from diag$(-,+,+,+)$ to diag$(+,-,-,-)$.

The double copy relation at the classical level can be surprisingly realized in a non-perturbative way. As mentioned in the introduction chapter 1, it has been shown that some exact solutions of the Einstein equation can be obtained by corresponding solutions of Yang-Mills theory.

The classical YM theory is usually studied as a toy model for gravity, but it is also important by itself. One example is that the equations of motion of classical YM theory describe the dynamics of the quark-gluon plasma, which is believed to be the predominant phase of matters before the entire universe was formed [101–103]. In particular, for describing high-energy heavy ions collisions, the gluon field is also treated classically as a first approximation [104–107].

In this chapter, we are interested in extending the Kerr-Schild double copy to the case of a probe particle moving in the static Yang-Mills and gravitational backgrounds. In particular, we will examine the dynamics of a test charge in the $\sqrt{\text{Schw}}$ and the equatorial plane of the $\sqrt{\text{Kerr}}$ background by directly solving the equations of motion. We characterize the orbits and find they feature elliptic, hyperbolic, and plunge behavior just as in the gravity background. More importantly, we reveal a double copy relation between the conserved quantities in the gauge and gravity theory, which enables us to fully recover geodesic equations for Schwarzschild and Kerr from that of $\sqrt{\text{Schw}}$ and $\sqrt{\text{Kerr}}$, respectively. Interestingly, the map works naturally for both bound and unbound orbits, which is beyond scattering amplitude double copy that only accounts for unbound cases.

We will introduce the double copy of conserved charges in section 4.1. This is followed by the detailed analysis of the trajectories of a test charge in $\sqrt{\text{Schw}}$ background in both massive and massless cases in section 4.2. We also show the double copy to Schwarzschild geodesics explicitly. In section 4.3, we perform similar calculations in the $\sqrt{\text{Kerr}}$ field but focus only on the equatorial plane.

4.1 Conserved Charges and the Double Copy

In this section, we derive the relation between the conserved charges of a test charged particle in the YM potential and the corresponding charges for a probe particle in the Kerr-Schild gravitational background. Let us first consider a point particle of mass m and color charge c moving in a YM background $A^a_\mu(x)$. Its dynamics is governed by the worldline action as (3.28). For convenience, we here rewrite the Lagrangian,

$$L^{\mathrm{YM}} = \frac{\bar{g}_{\mu\nu}v^\mu v^\nu}{2e} + \frac{em^2}{2} - i\psi^\dagger \frac{\mathrm{d}\psi}{\mathrm{d}\tau} + g\, c^a v^\mu A^a_\mu(x), \tag{4.1}$$

where $v^\mu(\tau) = \mathrm{d}x^\mu/\mathrm{d}\tau$ is the full four-velocity.[1] Note that (4.1) is valid for both $m > 0$ and $m = 0$. For massive particles, we set $e(\tau) = 1/m$ as in (3.30). In the massless case, we simply take $e(\tau) = 1$, giving the affine parametrization as explained in section 1.6. The constraint on the velocity is

$$\bar{g}_{\mu\nu}v^\mu v^\nu = \lambda \qquad \begin{cases} \lambda = 1 & \text{for} \quad m > 0 \\ \lambda = 0 & \text{for} \quad m = 0. \end{cases} \tag{4.2}$$

We consider a static YM field of the Kerr-Schild "single copy" form (1.50). For a charged test particle moving in this background we crucially require the coupling constant to be small enough not to affect the gauge field configuration. Let us now focus on the conserved quantities of the test particle. Suppose we have a cyclic coordinate ξ, which doesn't appear explicitly in the Lagrangian. From Noether's theorem, we know that the corresponding conserved charge is[2]

$$p^{\mathrm{YM}}_\xi = \frac{\partial L^{\mathrm{YM}}}{\partial v^\xi} = \frac{\partial v^\mu}{\partial v^\xi}\left(\frac{\bar{g}_{\mu\nu}v^\nu}{e} + \frac{g^2}{4\pi}c^a\tilde{c}^a\varphi(x)k_\mu(x)\right). \tag{4.3}$$

Likewise, the Lagrangian of the point mass in the corresponding double copy gravitational Kerr-Schild background reads

$$L^{\mathrm{GR}} = \frac{(\bar{g}_{\mu\nu} + \kappa h_{\mu\nu})v^\mu v^\nu}{2e} + \frac{em^2}{2}, \tag{4.4}$$

where we have separated the deviation from the Minkowskian space

$$\kappa h_{\mu\nu} = -2\, GM\varphi(x)k_\mu(x)k_\nu(x). \tag{4.5}$$

Again, setting the einbein $e(\tau) = 1/m$ for massive particles and $e(\tau) = 1$ for massless particles gives us the relativistic constraint

$$(\bar{g}_{\mu\nu} + \kappa h_{\mu\nu})v^\mu v^\nu = \lambda \qquad \begin{cases} \lambda = 1 & \text{for} \quad m > 0 \\ \lambda = 0 & \text{for} \quad m = 0. \end{cases} \tag{4.6}$$

It is clear that ξ is also a cyclic coordinate for L^{GR}, so we have a conserved charge for the point mass

$$p^{\mathrm{GR}}_\xi = \frac{\partial L^{\mathrm{GR}}}{\partial v^\xi} = \frac{\partial v^\mu}{\partial v^\xi}\left(\frac{\bar{g}_{\mu\nu}v^\nu}{e} - \frac{2GM}{e}\varphi k_\nu v^\nu k_\mu\right). \tag{4.7}$$

[1]Note that the definition of $v^\mu(\tau)$ is different from the background value v^μ defined in worldline quantum field theories in chapter 3.

[2]See [108, 109] for alternative approaches on how to derive the conserved charges.

Comparing eq. (4.3) and (4.7), we can derive the correspondence rules for Kerr-Schild double copy,

$$\frac{g^2}{4\pi} \to 2G \qquad \tilde{c}^a \to -Mk_\mu. \tag{4.8}$$

The relation mapping the conserved charges in Yang-Mills to those in gravity background follows

$$c^a \to \frac{v^\mu}{e} \quad \text{so that} \quad C := c \cdot \tilde{c} \to -\frac{M}{e} k \cdot v. \tag{4.9}$$

We note that the double copy map works for both $C > 0$ and $C < 0$, corresponding to repulsive and attractive forces, respectively. Nevertheless, in the analysis of solutions of the equations of motion, we will focus on the case $C < 0$ to resemble gravity, where the interaction is always "attractive"(see Fig. 4.1).

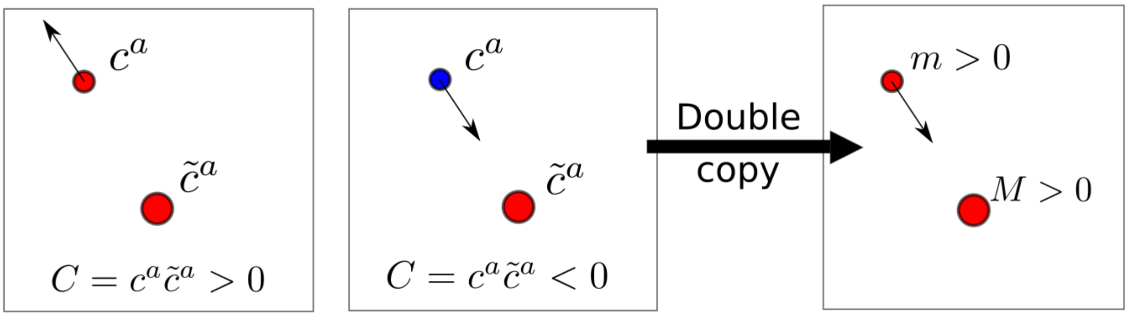


Figure 4.1: For a massive charged particle, we can have both attractive and repulsive gauge theory forces depending on the sign of the charges. This suggests focusing on the case $C < 0$, because masses in gravity are always positive.

In the case where the dynamics are integrable, knowing the conserved charges is sufficient to solve the equations of motion completely. In particular, this is true for $\sqrt{\text{Schw}}$ and equatorial orbits in $\sqrt{\text{Kerr}}$. In the following sections, we will apply (4.9) to obtain the conserved energy and angular momentum for a probe particle moving in the Schwarzschild background and on the equatorial plane of the Kerr background.

4.2 Test Charge in Coulomb-like Background

The Euler–Lagrange equations of (4.1) give us Wong's equations

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\nu\rho} v^\nu v^\rho = egc^a F^{a,\mu}{}_\nu v^\nu \tag{4.10}$$

$$\frac{\mathrm{d}c^a}{\mathrm{d}\tau} = gf^{abc} v^\mu A^b_\mu c^c(\tau), \tag{4.11}$$

where $\Gamma^\mu{}_{\nu\rho}$ is the Christoffel symbol for spherical coordinates, and e is chosen to be a constant. For $\sqrt{\text{Schw}}$ background in spherical coordinates (1.53), the field strength is

$$F^a{}_{rt} = -F^a{}_{tr} = -\frac{g}{4\pi}\frac{\tilde{c}^a}{r^2}, \tag{4.12}$$

with all other components vanishing. Thanks to the spherical symmetry of the problem, we can restrict our analysis to the $x-y$ plane by setting $\theta=\pi/2$ and $d\theta/d\tau=0$. Then the $\theta-$component of (4.10) is simply

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)=0 \quad \rightarrow \quad L:=r^2v^\phi, \tag{4.13}$$

which corresponds to the conservation of the z-component of the angular momentum L. This is universal for both massive and massless cases.

4.2.1 Massive Probe

Let us now focus on the specific case with a massive probe. Setting $e=1/m$ and restricting to the $\theta=\pi/2$ plane, we can rewrite Wong's equations as

$$\begin{aligned}\frac{\mathrm{d}v^r(\tau)}{\mathrm{d}\tau}-\frac{L^2}{r(\tau)^3}&=\frac{g^2}{4\pi m}\frac{c^a(\tau)\tilde{c}^a}{r(\tau)^2}v^t(\tau)\\ \frac{\mathrm{d}v^t(\tau)}{\mathrm{d}\tau}&=\frac{g^2}{4\pi m}\frac{c^a(\tau)\tilde{c}^a}{r(\tau)^2}v^r(\tau)\\ \frac{\mathrm{d}c^a(\tau)}{\mathrm{d}\tau}&=\frac{g^2}{4\pi}f^{abc}v^t(\tau)\frac{\tilde{c}^bc^c(\tau)}{r(\tau)},\end{aligned} \tag{4.14}$$

where we have made manifest the explicit dependence on the proper time τ. A crucial ingredient in solving the equations of motion is to observe that the scalar product of the two color vectors $C:=c^a(\tau)\tilde{c}^a$ is always conserved

$$\frac{\mathrm{d}C}{\mathrm{d}\tau}=\frac{g^2}{4\pi}f^{abc}v^t(\tau)\frac{\tilde{c}^a\tilde{c}^bc^c(\tau)}{r(\tau)}=0. \tag{4.15}$$

In the following, we will consider color charges of opposite signs so that the force is attractive: therefore $C<0$. Another significant conserved charge is the energy which can be defined from the $t-$component of Wong's equation

$$\frac{\mathrm{d}v^t(\tau)}{\mathrm{d}\tau}=-\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{g^2}{4\pi m}\frac{c^a(\tau)\tilde{c}^a}{r(\tau)}\right] \qquad \rightarrow \qquad h:=v^t+\frac{\alpha}{m}\frac{C}{r}, \tag{4.16}$$

where for conciseness we have defined $\alpha=g^2/4\pi$. The energy (4.16) and angular momentum charge (4.13) could also be derived directly from the Lagrangian approach as (4.3).

Thanks to (4.15), the non-Abelian $\sqrt{\text{Schw}}$ potential problem can be effectively reduced to the Abelian Coulomb potential, where the strength of the potential is determined by C. Even though the structure of the solution has been fully known in the Abelian case since a long time ago [110], we would like to display it here to show the features of the orbits.

Using the $r-$component of the equations of motion, we have

$$\frac{\mathrm{d}^2r(\tau)}{\mathrm{d}\tau^2}-\frac{L^2}{r(\tau)^3}=\frac{\alpha}{m}\frac{C}{r(\tau)^2}\left(h-\frac{\alpha}{m}\frac{C}{r(\tau)}\right), \tag{4.17}$$

and changing variable to $u:=\frac{1}{r}$ as a function of ϕ

$$\frac{\mathrm{d}^2u(\phi)}{\mathrm{d}\phi^2}+u(\phi)=-\frac{\alpha C}{mL^2}\left(h-\frac{\alpha Cu(\phi)}{m}\right), \tag{4.18}$$

where we have used the simple relation

$$\frac{\mathrm{d}u}{\mathrm{d}\phi} = -\frac{1}{L}\frac{\mathrm{d}r}{\mathrm{d}\tau}. \tag{4.19}$$

Moreover, $u(\phi)$ is constrained by the relativistic condition $\bar{g}_{\mu\nu}v^\mu v^\nu = 1$ which effectively reduces the degrees of freedom to the ones of a first order differential equation. If we define the critical value of the angular momentum

$$L_{\text{crit}} = -\frac{\alpha C}{m}, \tag{4.20}$$

then (4.18) can be rewritten as

$$\frac{\mathrm{d}^2 u(\phi)}{\mathrm{d}\phi^2} + u(\phi) = \frac{L_{\text{crit}}}{L^2}\left(h + L_{\text{crit}} u(\phi)\right), \tag{4.21}$$

while $\bar{g}_{\mu\nu}v^\mu v^\nu = 1$ gives

$$\left(\frac{\mathrm{d}u(\phi)}{\mathrm{d}\phi}\right)^2 = \frac{1}{L^2}\left[(h + L_{\text{crit}}u(\phi))^2 - L^2 u(\phi)^2 - 1\right]. \tag{4.22}$$

The analytic solution of the differential equation (4.21) for $L < L_{\text{crit}}$ is

$$u^{(\pm)}(\phi) = \frac{B_1 L \sinh\left(\frac{\phi\sqrt{L_{\text{crit}}^2 - L^2}}{L}\right)}{\sqrt{L_{\text{crit}}^2 - L^2}} + \frac{\left(B_2^{(\pm)}\left(L^2 - L_{\text{crit}}^2\right) - hL_{\text{crit}}\right)\cosh\left(\frac{\phi\sqrt{L_{\text{crit}}^2 - L^2}}{L}\right) + hL_{\text{crit}}}{L^2 - L_{\text{crit}}^2}, \tag{4.23}$$

where

$$B_1 = u(\phi)\Big|_{\phi=0} \qquad B_2 = \frac{du(\phi)}{d\phi}\Big|_{\phi=0}$$
$$B_2^{(\pm)} = \pm\frac{1}{L}\sqrt{(h + B_1 L_{\text{crit}})^2 - B_1^2 L^2 - 1}. \tag{4.24}$$

The last equation is directly deduced from eq. (4.23). A similar result holds for $L > L_{\text{crit}}$ by using analytic continuation arguments

$$u^{(\pm)}(\phi) = \frac{B_1 L \sin\left(\frac{\phi\sqrt{L^2 - L_{\text{crit}}^2}}{L}\right)}{\sqrt{L^2 - L_{\text{crit}}^2}} + \frac{\left(B_2^{(\pm)}\left(L^2 - L_{\text{crit}}^2\right) - hL_{\text{crit}}\right)\cos\left(\frac{\phi\sqrt{L^2 - L_{\text{crit}}^2}}{L}\right) + hL_{\text{crit}}}{L^2 - L_{\text{crit}}^2}. \tag{4.25}$$

Instead, the critical case $L = L_{\text{crit}}$ gives

$$u^{(\pm)}(\phi) = B_2^{(\pm)} + B_1\phi + \frac{h\phi^2}{2L_{\text{crit}}}, \tag{4.26}$$

where B_1 and $B_2^{(\pm)}$ are again given by (4.24). The presence of a critical value of the angular momentum, which is crucial for the classification of the orbits, is related to the relativistic nature of the problem (see the analysis in [111]).

It is convenient to analyze the nature of the orbits by looking at the zeros of the potential, which will be instructive in preparation for the next section. We have

$$\begin{aligned}\left(\frac{du(\phi)}{d\phi}\right)^2 =&\frac{1}{L^2}\left[(h+L_{\text{crit}}u(\phi))^2-L^2u(\phi)^2-1\right]\\ =&\frac{L_{\text{crit}}^2-L^2}{L^2}(u(\phi)-u_+)(u(\phi)-u_-)\\ u_\pm :=&\frac{hL_{\text{crit}}\pm\sqrt{(h^2-1)\,L^2+L_{\text{crit}}^2}}{L^2-L_{\text{crit}}^2}.\end{aligned} \tag{4.27}$$

where $\frac{du}{d\phi}\big|_{u=u_+}=\frac{du}{d\phi}\big|_{u=u_-}=0$. From this we can characterize the possible orbits.

- Elliptic bound orbits (see Fig. 4.2a) which require two positive roots $u_\pm>0$ with $\frac{d^2u}{d\phi^2}\big|_{u=u_+}<0$ and $\frac{d^2u}{d\phi^2}\big|_{u=u_-}>0$, that is

$$L_{\text{crit}}<|L|<\frac{1}{\sqrt{1-h^2}}L_{\text{crit}}\qquad 0<|h|<1. \tag{4.28}$$

- Circular orbits (see Fig. 4.2b), which correspond to $u_+=u_-=u_*$ with $\frac{d^2u}{d\phi^2}\big|_{u=u_*}=0$, i.e.

$$L=\pm\frac{1}{\sqrt{1-h^2}}L_{\text{crit}}\qquad 0<|h|<1, \tag{4.29}$$

  where the radius of such orbits is

$$r_*=\frac{1-h^2}{hL_{\text{crit}}}. \tag{4.30}$$

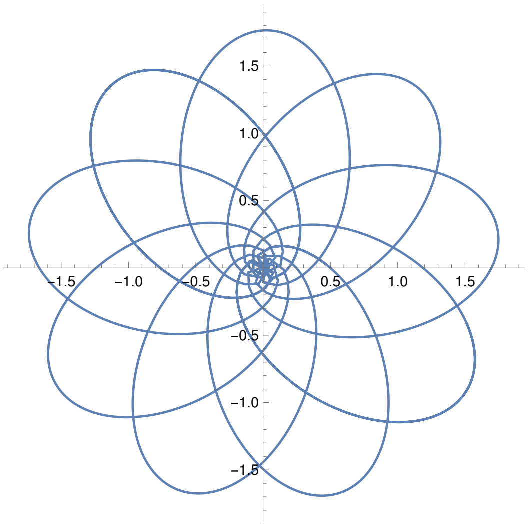
- Hyperbolic-type unbound orbits where the probe escapes to infinity (see Fig. 4.2c), which require just one root to be real and positive $u_->0$ with $\frac{d^2u}{d\phi^2}\big|_{u=u_-}<0$,

$$\begin{aligned}&|L|>L_{\text{crit}} &&h\geq 1\\ &|L|\leq L_{\text{crit}} &&h\geq 1\qquad B_2=B_2^{(+)}.\end{aligned} \tag{4.31}$$

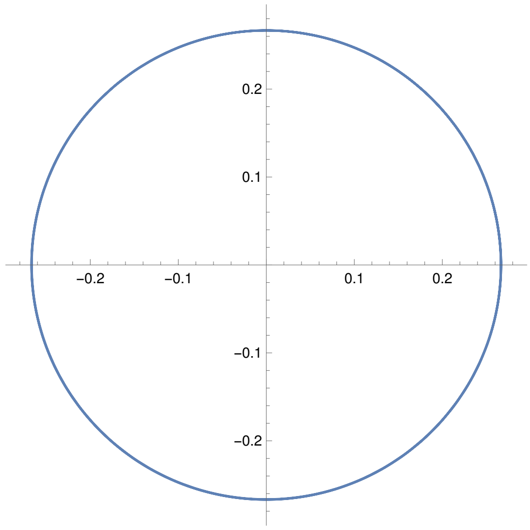
- Plunge-type orbits for the probe particle (see Fig. 4.2d), provided $u_\pm\leq 0$ for $h>1$ or $u_->0>u_+$ for $0<h<1$,

$$\begin{aligned}&|L|\leq L_{\text{crit}} &&0<|h|<1\\ &|L|\leq L_{\text{crit}} &&h\geq 1\qquad B_2=B_2^{(-)}.\end{aligned} \tag{4.32}$$

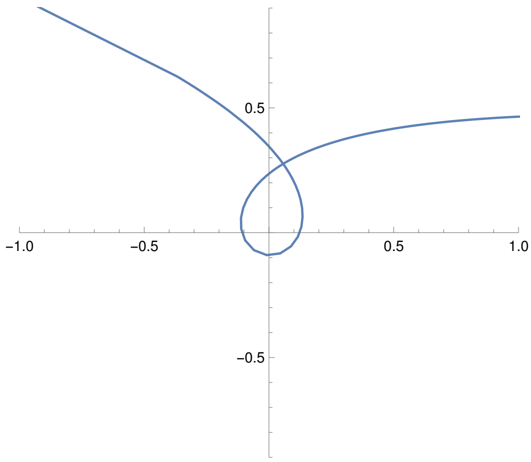
We observe that for $|h|<1$ we always have bound orbits, i.e., the massive particle cannot escape to time-like infinity. This is not surprising since in the limit $r\to\infty$, from eq. (4.16) we have $h=v^t$, which has to be greater than 1 because of causality. For the same reason, hyperbolic orbits are allowed only when $h\geq 1$.



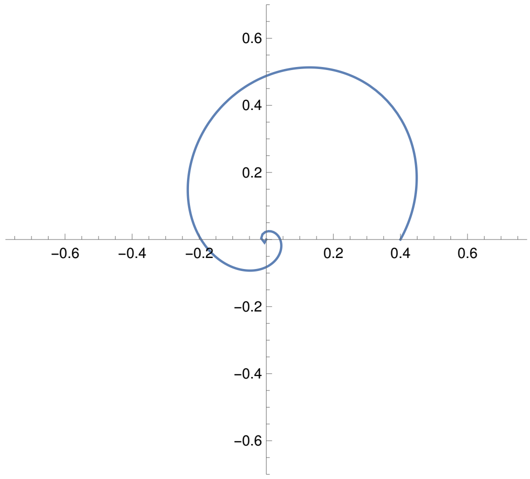
(a) Elliptic orbit



(b) Circular orbit



(c) Hyperbolic-type orbit



(d) Plunge orbit

Figure 4.2: Types of orbits for a massive charged particle in the $\sqrt{\text{Schw}}$ potential.

4.2.2 Massless Probe

In the massless case, we choose $e = 1$. Compared to the massive case, we effectively only need to make the replacement

$$\alpha/m \to \alpha, \qquad \tau \to \tau' \tag{4.33}$$

in (4.21) to get the new radial equation of motion for the massless charged particle

$$\frac{d^2u(\phi)}{d\phi^2} + u(\phi) = \frac{L'_{\text{crit}}}{(L')^2}\left(h' + L'_{\text{crit}}u(\phi)\right), \tag{4.34}$$

with

$$h' := v^t + \alpha\frac{C}{r} \qquad L' := r^2 v^\phi \qquad L'_{\text{crit}} := -\alpha C. \tag{4.35}$$

The constraint equation $\bar{g}_{\mu\nu}v^\mu v^\nu = 0$ gives,

$$\left(\frac{du(\phi)}{d\phi}\right)^2 = \frac{1}{(L')^2}\left[\left(h' + L'_{\text{crit}}u(\phi)\right)^2 - (L')^2 u(\phi)^2\right], \tag{4.36}$$

and the explicit solution will be given by (4.23), (4.25) and (4.27), but with the new constraint equation (4.36) in place of (4.22).

It is parallel to the massive case to characterize the solutions.

- Circular orbits for

$$L' = L'_{\text{crit}} \qquad h' = 0, \tag{4.37}$$

  with the surprising feature that the radius of the orbit is not constrained.

- Hyperbolic-type unbound orbits for

$$\begin{aligned} &|L'| > L'_{\text{crit}} && \text{or} \\ &|L'| \leq L'_{\text{crit}} && B_2 = B_2^{(+)}. \end{aligned} \tag{4.38}$$

- Plunge behaviour for

$$|L'| \leq L'_{\text{crit}} \qquad B_2 = B_2^{(-)}. \tag{4.39}$$

These type of orbits are well-represented by the pictures in Fig. 4.2a, 4.2c and 4.2d respectively. We would like to note that here, compared to the massive case, elliptic orbits are not allowed, and circular orbits are allowed only in the very degenerate limit $h' = 0$.

4.2.3 To Schwarzschild Geodesics

With the equations (4.13) and (4.16), we are now ready to derive the corresponding conserved quantities of a (massive or massless) probe moving on the equatorial plane in a Schwarzschild black hole background. Applying (4.8) and (4.9) to

$$L^{\sqrt{\text{Schw}}} = r^2 v^\phi \tag{4.40}$$

$$h^{\sqrt{\text{Schw}}} = v^t + \alpha e\frac{C}{r}, \tag{4.41}$$

we get

$$L^{\text{Schw}} = r^2 v^\phi \tag{4.42}$$

$$h^{\text{Schw}} = \left(1 - \frac{2GM}{r}\right)v^t - \frac{2GM}{r}v^r. \tag{4.43}$$

The constraint (4.6) can be written in terms of the conserved charges as

$$(v^r)^2 = \lambda + (h^{\text{Schw}})^2 - \left(1 + \frac{(L^{\text{Schw}})^2}{r^2}\right)\left(1 - \frac{2GM}{r}\right). \tag{4.44}$$

Since the dynamics is integrable, with (4.42), (4.43) and (4.44), one can fully solve the geodesic problem in Schwarzschild. In particular, this implies that the impulse and other observables in the probe limit are completely determined by the double copy map [112].

4.3 Test Charge in Spinning Background

The $\sqrt{\text{Kerr}}$ background is given in (1.57). It is more convenient to work in the spheroidal coordinates (t, r, θ, ϕ)

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin(\theta)\cos(\phi) \\ y &= \sqrt{r^2 + a^2}\sin(\theta)\sin(\phi) \\ z &= r\cos(\theta), \end{aligned} \tag{4.45}$$

in which the flat metric is

$$\bar{g}_{\mu\nu} = \mathrm{diag}\left(1, -\frac{r^2 + a^2\cos^2(\theta)}{r^2 + a^2}, -r^2 - a^2\cos^2(\theta), -(a^2 + r^2)\sin^2(\theta)\right). \tag{4.46}$$

In the limit where $a \to 0$, this recovers the standard spherical coordinates. Without loss of generality, we will always assume $a > 0$. The components of the gauge field in these coordinates are

$$\begin{aligned} A_t^a &= \frac{g}{4\pi}\frac{r\tilde{c}^a}{r^2 + a^2\cos^2(\theta)} \qquad & A_r^a &= \frac{g}{4\pi}\frac{r\tilde{c}^a}{r^2 + a^2} \\ A_\phi^a &= -\frac{g}{4\pi}\frac{r\tilde{c}^a}{r^2 + a^2\cos^2(\theta)}a\sin^2(\theta) \qquad & A_\theta^a &= 0. \end{aligned} \tag{4.47}$$

For simplicity, hereafter we will focus on equatorial orbits by setting $\theta = \pi/2$. We would like to stress that the problem can be solved in full generality, but the complexity is higher for non-equatorial orbits, exactly like for Kerr geodesics. The non-vanishing components of the field strength are

$$F_{rt}^a = -F_{tr}^a = -\frac{g}{4\pi}\frac{\tilde{c}^a}{r^2} \qquad F_{r\phi}^a = -F_{r\phi}^a = a\frac{g}{4\pi}\frac{\tilde{c}^a}{r^2}. \tag{4.48}$$

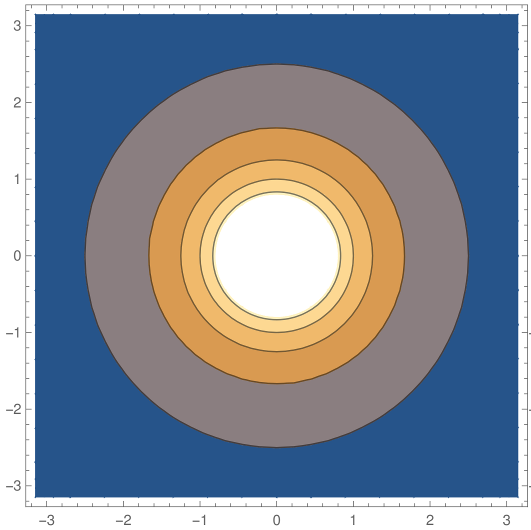
In the following, we will consider only orbits that lie outside the ring singularity at $x^2+y^2 = a^2$ on the equatorial plane where the spheroidal coordinates (4.47) are always well-defined.

In order to develop an intuitive image of the $\sqrt{\text{Kerr}}$ field, we would like to show here the structure of the time component of the potential A_t, projected along the $x - z$ plane (there is always an azimuthal symmetry). While for $a > 0$ at large distances from the singularity, A_t has an ellipsoidal shape, closer to the singularity line of width a the potential develops a dipole-type configuration.

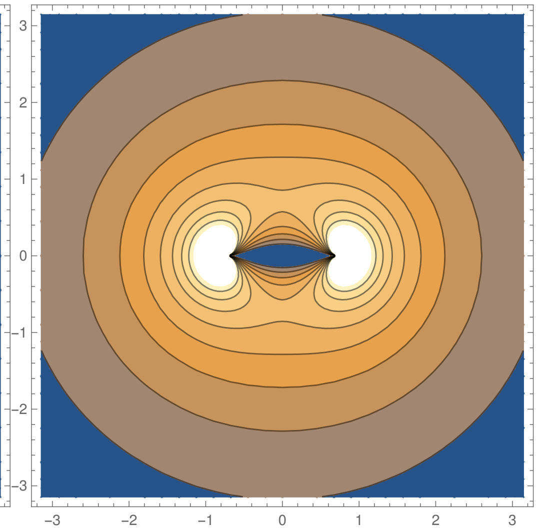
4.3.1 Massive Probe

For a massive probe, Wong's equations on the equatorial plane are

$$\begin{gathered} \frac{\mathrm{d}v^t}{\mathrm{d}\tau} = \frac{\alpha}{m}\frac{C}{r^2}v^r, \\ \frac{\mathrm{d}v^r}{\mathrm{d}\tau} - \frac{1}{r}(a^2 + r^2)\left(v^\phi\right)^2 + \frac{1}{r}\frac{a^2}{a^2 + r^2}(v^r)^2 = \frac{\alpha}{m}\frac{C}{r^4}\left[\left(v^t - av^\phi\right)(a^2 + r^2)\right], \\ \frac{\mathrm{d}v^\theta}{\mathrm{d}\tau} = 0, \\ \frac{\mathrm{d}v^\phi}{\mathrm{d}\tau} + 2\frac{r}{r^2 + a^2}v^r v^\phi = a\frac{\alpha}{m}\frac{C}{r^2(r^2 + a^2)}v^r. \end{gathered} \tag{4.49}$$



(a) a=0



(b) a=0.7

Figure 4.3: Regions of the same color on the $x - z$-plane (with z pointing upwards) are divided by lines of constant A_t.

As in the Coulomb potential case, there is a notion of conserved energy and angular momentum

$$h := v^t + \frac{\alpha}{m}\frac{C}{r} \tag{4.50}$$

$$L := (r^2 + a^2)v^\phi + a\frac{\alpha}{m}\frac{C}{r}. \tag{4.51}$$

In particular, the angular momentum now also includes contributions from the parameter a, which is perfectly analogous to the Kerr case. At this point, we can use those conserved charges to derive the equation of motion for $u(\tau)$ using (4.49). We find

$$\begin{aligned}\frac{\mathrm{d}v^r}{\mathrm{d}\tau} - \frac{(Lr + a(L_{\text{crit}} - rv^r))(Lr + a(L_{\text{crit}} + rv^r))}{r^3(a^2 + r^2)}& \\ = -\frac{L_{\text{crit}}}{r^4}[a(ah - L) + r(L_{\text{crit}} + hr)].&\end{aligned} \tag{4.52}$$

The constrained equation $\bar{g}_{\mu\nu}v^\mu v^\nu = 1$ gives

$$\begin{aligned}(v^r)^2 &= \left(1 + \frac{a^2}{r^2}\right)(h - V_+)(h - V_-) - \left(1 + \frac{a^2}{r^2}\right) \\ V_\pm &:= \frac{1}{r}\left[-L_{\text{crit}} \pm \frac{|Lr + aL_{\text{crit}}|}{\sqrt{r^2 + a^2}}\right].\end{aligned} \tag{4.53}$$

It is worth noticing here that the leading term in $1/r$ on the RHS of (4.52) is proportional to a and $ah - L$, which is a feature shared also by Kerr equatorial orbits [113]. In the spinless limit $a \to 0$, the solution reduces to the Coulomb non-abelian potential we already considered in the last section because

$$V_\pm \overset{a=0}{\to} \frac{1}{r}(-L_{\text{crit}} \pm |L|). \tag{4.54}$$

Regarding the case where $L = ah$, an analytic solution for τ as a function of r exists (similarly to Kerr black hole equatorial geodesic), but we will not display it here.

A key question is whether circular orbits exist at all. Furthermore, if so, what are the corresponding values for the energy and the angular momentum in these cases? The condition for the existence of circular orbits at $r = r_*$ (meaning that the radius is $\sqrt{r_*^2 + a^2}$) is given by the common solution of

$$v^r|_{r=r_*} = 0 \qquad \left.\frac{\mathrm{d}v^r}{\mathrm{d}r}\right|_{r=r_*} = 0. \tag{4.55}$$

With some algebra we can reduce this system of equations to

$$\begin{aligned} h^2 &= 1 - \frac{L_{\text{crit}}}{r_*^3}\left(ax + hr_*^2\right) \qquad x := L - ah \\ \left(a^2 - x^2\right)^2 &= -\frac{L_{\text{crit}}}{r_*^3}\left(a^2 + r_*^2\right)\left(x^2 + r_*^2\right)\left(4ax - L_{\text{crit}} r_*\right), \end{aligned} \tag{4.56}$$

where the last equation is a quartic polynomial in x. For every $a > 0$ there are two distinct real (and therefore other two complex) solutions for (4.56) (see Appendix A for a proof.). We will call the real roots x_1 and x_2, and we order them as $x_1 > x_2$. The value of the energy and the angular momentum for such circular orbits is given by (4.56), i.e., explicitly

$$\begin{aligned} h_{1,2}^{\pm} &= \frac{1}{2r_*}\left(-L_{\text{crit}} \pm \sqrt{L_{\text{crit}}^2 + 4r_*^2 - 4\frac{a}{r_*}L_{\text{crit}} x_{1,2}}\right) \\ L_{1,2}^{\pm} &= x_{1,2} + ah_{1,2}^{\pm}, \end{aligned} \tag{4.57}$$

where only (h_1^+, L_1^+) always satisfies the causality constraint. As we will see later, this solution will indeed be related to stable circular orbits.

In order to study the general case, we need to analyze the nature of the roots of the right-hand side of the constraint equation (4.53). Specifically, we have

$$\begin{aligned} (v^r)^2 &= \frac{1}{r^3}\mathcal{P}(r) \\ \mathcal{P}(r) &:= \left(h^2 - 1\right) r^3 + 2hL_{\text{crit}} r^2 \\ &\quad + \left(a^2\left(h^2 - 1\right) + L_{\text{crit}}^2 - L^2\right) r + 2aL_{\text{crit}}(ah - L) \end{aligned} \tag{4.58}$$

which defines a third-order polynomial $\mathcal{P}(r)$, to which we can apply the tools developed in Appendix A in order to understand the nature of the roots. By computing the reduce discriminant $\Delta_R(\mathcal{P}(r))$ (see (A.5)), we can establish whether

- $\mathcal{P}(r)$ has three simple real roots ($\Delta_R(\mathcal{P}(r)) > 0$)
- $\mathcal{P}(r)$ has one simple real root ($\Delta_R(\mathcal{P}(r)) < 0$)
- $\mathcal{P}(r)$ has a double or triple root ($\Delta_R(\mathcal{P}(r)) = 0$)

and by using Descartes' rule of signs, we can also understand the number of positive or negative roots. We note that in the case $\Delta_R(\mathcal{P}(r)) = 0$ there are circular orbits with radius given by solving eq. (4.56). However, $\mathcal{P}(r)$ by itself is a polynomial in h of degree eight, which is impossible to solve analytically. Therefore, we can only qualitatively analyze the real solutions of $\mathcal{P}(r) = 0$. We find that the properties of the solution depend on the values of L_{crit} and L. Specifically, we find exactly two cases:

- Case 1) $L_{\text{crit}} < a$: for a given value of L there are at most 4 real solutions for h, and only two of them can be positive. We denote them as h_A, h_B with $h_A < h_B$ when they exist.
- Case 2) $L_{\text{crit}} > a$: for a given value of L, there are either 2 or 4 real solutions for h. One of these real solutions is h_1^+, which is defined by (4.57).

In addition, as discussed before, $h = L/a$ represents another critical value of the energy since having $h > L/a$ or $h < L/a$ will change of the sign of the constant term in the polynomial $\mathcal{P}(r)$. A detailed analysis of the orbits shows that we can have the following cases

- Elliptic orbits for[3]

$$\begin{aligned} h_A < h < \min\{h_B, 1\} \qquad & L_{\text{crit}} < a \\ h_1^+ < h < \min\{L/a, 1\} \qquad & L_{\text{crit}} > a \end{aligned}$$

- Hyperbolic-type orbits for $h > 1$ in all cases
- Plunge behavior for $h > L/a$ in all cases

where h_A can be identified with the stable circular orbit value h_1^+, and it is understood that when two intervals overlap, we can have different types of orbits according to the boundary conditions (like the sign of the initial velocity or also the initial radial coordinate). We have represented the typical behavior of those solutions in Fig. 4.4a, Fig. 4.4c and Fig. 4.4d, where we have also highlighted in red the ring singularity at $r = 0$ where the gauge potential has a singular behavior.

An interesting limit is the one that correspond to marginally bound circular orbits with $h = 1$ (see Fig. 4.4b): in such case, we can find a simple analytic solution for the value of the radius and the charges

$$\begin{aligned} L_\pm\Big|_{h=1} &= a - \left(\sqrt{a} \pm \sqrt{L_{\text{crit}}}\right)^2 \\ r_{*,\pm}\Big|_{h=1} &= a \pm \sqrt{aL_{\text{crit}}}, \end{aligned} \tag{4.59}$$

where $r_{*,-}$ exists only when $L_{\text{crit}} < a$. Since our main goal is to connect the conserved charges on the gauge side with the ones on the gravity side, we leave a complete analytic analysis of the generic orbits for massive particles in the case $a > 0$ for a future study.

4.3.2 Massless Probe

Using (4.33), we can derive the corresponding equations of motion for massless charged test particle in the $\sqrt{\text{Kerr}}$ potential. In particular, the relativistic constraint equation becomes

$$(v^r)^2 = \left(1 + \frac{a^2}{r^2}\right)(h' - V'_+)(h' - V'_-), \tag{4.60}$$

where the potential in the massless case is

$$V'_\pm := \frac{1}{r}\left[-L'_{\text{crit}} \pm \frac{|L'r + aL'_{\text{crit}}|}{\sqrt{r^2 + a^2}}\right]. \tag{4.61}$$

[3]When $L_{\text{crit}} = a$, h_B becomes equal to L/a.

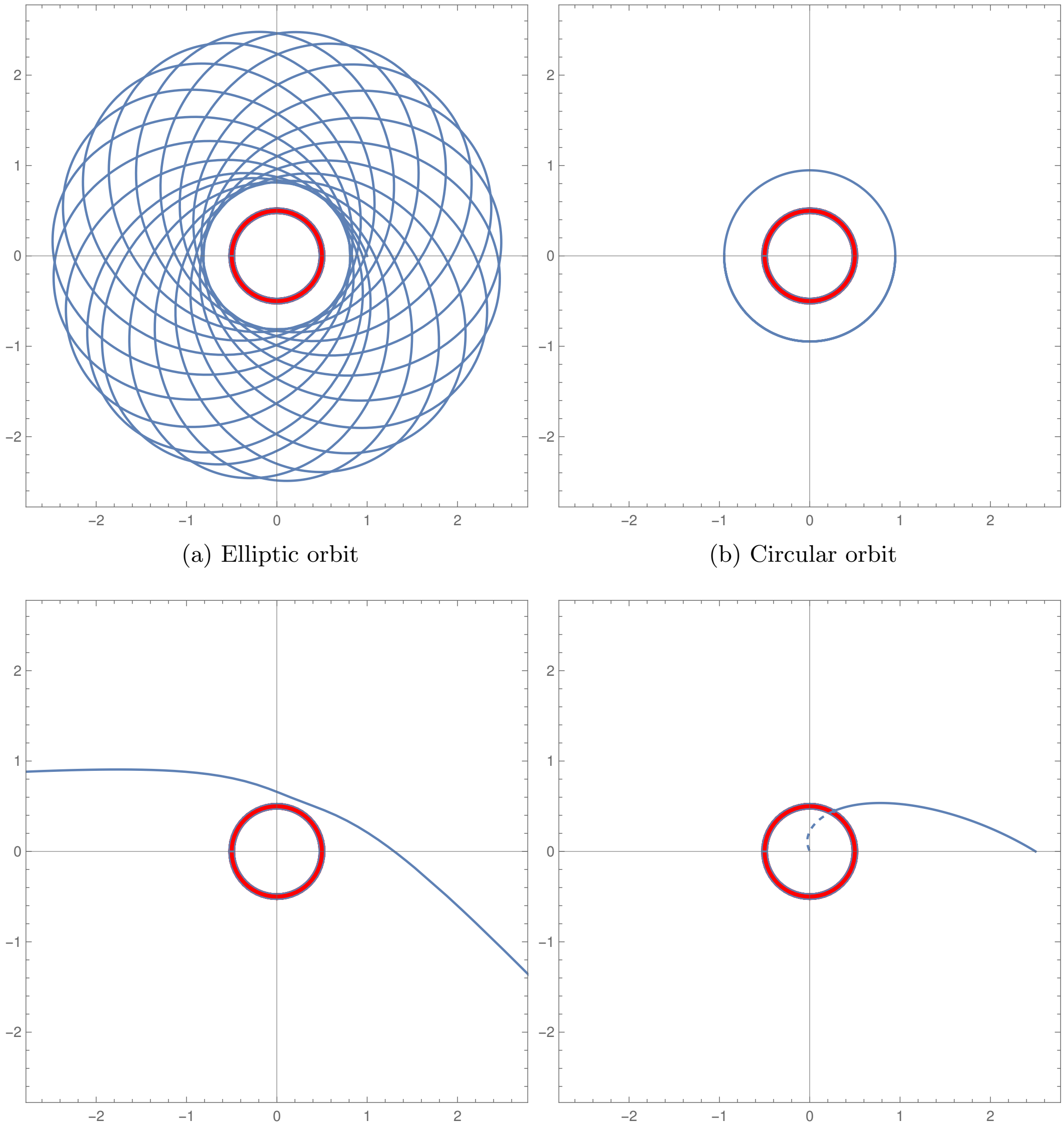


Figure 4.4: Types of orbits for a massive charged particle in the $\sqrt{\text{Kerr}}$ potential. The red ring represents the singularity of the field.

The conserved quantities read

$$h' := v^t + \alpha\frac{C}{r} \qquad L'_{\text{crit}} := -\alpha C \tag{4.62}$$

$$L' := (r^2 + a^2)v^\phi + \frac{\alpha a C}{r}. \tag{4.63}$$

Causality requires $v^t > 0$ and therefore the conserved energy has to satisfy $h' > -L'_{\text{crit}}/r$. Meanwhile, from (4.60) we know the region $V'_- < h' < V'_+$ is forbidden otherwise the right-hand side is negative. Since $V'_- \leq -L'_{\text{crit}}/r \leq V'_+$, the physically meaningful value of the energy is constrained as $h' \geq V'_+$. Taking the time-derivative of (4.60) we can derive

$$\frac{\mathrm{d}v^r}{\mathrm{d}\tau} = -\left(1 + \frac{a^2}{r^2}\right)\left[\frac{dV'_+}{dr}(h' - V'_-) + \frac{dV'_-}{dr}(h' - V'_+)\right] - \frac{a^2}{r^3}(h' - V'_+)(h' - V'_-) \tag{4.64}$$

and thus the condition to have circular orbits at $r = r_*$ is equivalent to

$$h'|_{r=r_*} = V'_+|_{r=r_*} \qquad \left.\frac{\mathrm{d}V'_+}{\mathrm{d}r}\right|_{r=r_*} = 0. \tag{4.65}$$

We find that a circular orbit requires the energy and angular momentum to satisfy

$$h'_\pm = \frac{a^2 L'_{\text{crit}}}{r_*^3}\left(1 \pm \sqrt{1 + \frac{r_*^2}{a^2}}\right) \tag{4.66}$$

$$L'_\pm = -\frac{a^3 L'_{\text{crit}}}{r_*^3}\left(1 + \frac{2r_*^2}{a^2} \pm \left(1 + \frac{r_*^2}{a^2}\right)^{3/2}\right). \tag{4.67}$$

As it is easy to show, in the vicinity of the singular ring $r \to 0$, we have

$$\lim_{r\to 0} V'_+(r) = \frac{L'}{a}. \tag{4.68}$$

A completed analysis for the orbits show that we can have

- Hyperbolic-type orbits for any positive value of the energy $h' > 0$.
- Plunge behaviour for $h' \geq L'/a$ both in the co-rotating case $L' > 0$ and in the in the counter-rotating case $L' < 0$.
- Elliptic orbits for $h'_- < h' < \min\{0, L'/a\}$ with $L' < L_{\text{crit}}$. h'_- is determined implicitly in terms of the minimum of V'_+.
- Stable circular orbits for $(h', L') = (h'_-, L'_-)$ with $L < L_{\text{crit}}$ and unstable circular orbits for $(h', L') = (h'_+, L'_+)$ with $L < -L_{\text{crit}}$.

where when two regions of the parameter space (h', L') overlap we can have different types of orbits according to the initial boundary conditions. This behavior of the massless probe particle is also (at least qualitatively) exemplified by the pictures in Fig. 4.4a, Fig. 4.4c, Fig. 4.4d and Fig. 4.4b of the previous section. Unlike null geodesics on the Kerr background, elliptic orbits are surprisingly allowed in the $\sqrt{\text{Kerr}}$ case and there are stable circular orbits.

4.3.3 To Kerr Geodesics

We can now obtain the conserved charges for a probe particle moving in the Kerr black hole background. In the Kerr-Schild form, we have

$$\varphi(x) = \frac{r}{a^2\cos^2(\theta) + r^2} \tag{4.69}$$

$$k_\mu = \left(1, \frac{r^2 + a^2\cos^2(\theta)}{a^2 + r^2}, 0, -a\sin^2(\theta)\right). \tag{4.70}$$

The conserved quantities in the Kerr spacetime is gained from (4.8) and (4.9), i.e. from

$$L^{\sqrt{\text{Kerr}}} = (r^2 + a^2)v^\phi + a\alpha e\frac{C}{r} \tag{4.71}$$

$$h^{\sqrt{\text{Kerr}}} = v^t + \alpha e\frac{C}{r}, \tag{4.72}$$

we get

$$L^{\text{Kerr}} = (r^2 + a^2)v^\phi - a\frac{2GM}{r}\left(\frac{r^2 v^r}{a^2 + r^2} - av^\phi + v^t\right) \tag{4.73}$$

$$h^{\text{Kerr}} = v^t - \frac{2GM}{r}\left(\frac{r^2 v^r}{a^2 + r^2} - av^\phi + v^t\right). \tag{4.74}$$

The null-like and time-like geodesics on the equatorial orbit can then be obtained by using these two charges and the four-velocity normalization condition (4.6).

4.4 Comments on geodesics double copy

The color-kinematics duality offers promising ideas to tackle complex problems in the gravitational setting, in particular regarding the two-body problem for two massive particles in general relativity. In the extreme limit where one mass is much bigger than the other, i.e., at leading order in the expansion in the mass ratio, the problem is equivalent to a light particle following geodesics in the background sourced by the other heavy particle. This setting allows solving the motion of the test particle exactly in some specific cases, which provides a way to examine and explore the double copy idea for classical solutions. In particular, for Schwarzschild and Kerr, their Kerr-Schild single copies correspond to a non-Abelian $1/r$ Coulomb-like potential ($\sqrt{\text{Schw}}$) and the potential generated by a rotating disk of charge ($\sqrt{\text{Kerr}}$), respectively [51].

We consider a test charged probe particle moving in the $\sqrt{\text{Schw}}$ background and the equatorial plane of the $\sqrt{\text{Kerr}}$ background, respectively. In each case, we explicitly solve Wong's equations in terms of the conserved energy h and the angular momentum L. particularly, we focus on the situation where the color charge $C = c^a\tilde{c}^a$ is negative so that we can correctly reproduce similar orbits with gravity, which are always attractive. We can then extend the Kerr-Schild double-copy to derive a mapping between conserved charges of a probe particle in the YM and the gravitational background. Specifically, the map (4.9) replaces the color charge of the test particle by its momentum in the spirit of color-kinematics duality. This allows us not only to fully recover the geodesic equations for Schwarzschild and Kerr but also provides the bridge with the perturbative double copy prescription for charged particles introduced by Goldberger and Ridgway to relate the gluon and the graviton radiative field [96]. In particular, while the double copy was initially used for scattering amplitudes and thus for unbound-like orbits, our mapping applies naturally also for bound problems [114]. The reason is that explicit solutions of equations of motion do have a natural analytic continuation in terms of the conserved charges [114–117].

The $\sqrt{\text{Schw}}$ and $\sqrt{\text{Kerr}}$ potential are of great interest on their own, in particular, to better understand the YM dynamics in a non-perturbative setting. Indeed, stability of these types of potentials was investigated by Mandula et al. a long time ago concerning the confinement mechanism [118–121]. For our work, we keep the coupling constant small enough so that the probe particle does not affect the gauge background. For the $\sqrt{\text{Schw}}$ case we have found an analytical solution for any orbit, and for the $\sqrt{\text{Kerr}}$ case, we have qualitatively discussed the behavior of the probe particle moving in equatorial orbits. For both potentials, we have found that massive test particles can move in elliptic, circular, hyperbolic-type, or plunge orbits depending on the values of the conserved charges. For massless particles, the situation is similar, but with a surprise: while elliptic orbits are not allowed, and circular orbits become unstable in $\sqrt{\text{Schw}}$, there are instead elliptic and stable circular orbits for $\sqrt{\text{Kerr}}$. With this exception in mind, what is striking to us is the similarity between those solutions for both backgrounds and the time-like and null-like geodesics of Schwarzschild/Kerr [113]. This is evident both from intermediate stage calculations and from the explicit analytic results. Note that because of the complexity, we refrain from considering the geodesics of $\sqrt{\text{Kerr}}$ outside of the equatorial plane, but it will be interesting to examine our analysis of the most generic case.

Summarizing, the probe limit contains much information on full two-body problem in general relativity [122–124], and our results provide another indication that such data is entirely encoded in the simpler gauge theory dynamics via the double copy map. This is also supported by previous evidence coming from the derivation of the impulse [112] and the multipole [56] using double copy techniques. While we have considered only the leading order contribution in the so-called self-force expansion, it would be nice to understand whether double copy can help shed light on the higher-order terms in the expansion in the mass ratio.

Chapter 5

Three-body Effective Potential in General Relativity

This chapter is based on the published article "Three-Body Effective Potential in General Relativity at Second Post-Minkowskian Order and Resulting Post-Newtonian Contributions" [4], written in collaboration with Prof. Dr. Jan Plefka, Dr. Florian Loebbert, and Dr. Tianheng Wang.

In the study of gravitational-wave physics, researchers focus more on the two-body problem as it is the major event observed by experiments. One wonders if the interactions beyond two bodies, i.e., N-body interactions, are significant enough to be observed by future gravitational wave detectors. For this, we need to have high precision predictions of the motion of the N bodies, as well as the radiated gravitational wave. The next level of two bodies is, of course, three bodies, which is the main concern of this chapter.

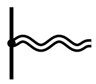
The motion of three bodies is intricate already in Newtonian gravity. Generic solution is impossible due to the chaotic nature of the problem. Only a few families of special solutions are known, for example, the figure-eight orbits of three equal-mass objects [125–127]. If one takes into account of special relativistic effects, the problem becomes even more involving. Moreover, in Newtonian gravity, the interaction of three bodies is merely the sum of the two-body interactions between any pair of two of the three objects, but in general relativity, we have genuine three-body interactions which involve all three bodies. which are absent in the two body limit.

In post-Newtonian (PN) limit, the conservative two-body potential is known up to 4PN [128–139]. Compared to that, few is known for the three-body case in gravity. The first correction to the Newtonian two-body interaction was derived by Einstein, Infeld and Hoffmann as early as in 1938 [140, 141]. Damour and Schäfer revisited the N-body problem in 1987 and found the 2PN contribution to the three-body interaction [142, 143]. For $N \geq 4$, even at 2PN there is no explicit expressions for the effective potential. In post-Minkowskian (PM) expansion, there is no genuine three-body contribution at 1PM [144]. In this chapter, we will present a formal form for the 2PM potential in (5.13), which depends on a one-loop three-point integral (5.14). At this order, three-body interaction is sufficient to have the complete N-body effective action. After the publication of our article [4], a novel method based on scattering amplitude techniques is proposed to obtain the 2PM effective Hamiltonian [145]. In present, we do not know much about the three-body potential beyond 2PM.

Let us briefly explain the approach we adopt in this chapter. Since we are mainly interested in the off-shell potential in this chapter, we will employ the effective worldline formalism. It starts with the same action (1.69) with WQFT introduced in section 3.1, but we do not expand around a straight line background and consequently we do not need to integrate out the worldline fluctuations. Instead, we will simply consider weak gravity around Minkowskian spacetime as usual and integrate only over the exchanged gravitons. This way can give us a non-local-in-time effective PM potential, in order to verify our calculation, we also want to compare with known PN results. Therefore, we will perform a slow-velocity expansion to the relativistic PM effective action. We then present a method to obtain PN contributions from the 2PM action provided that one knows how the evaluate the critical family of integrals (5.44).

5.1 Effective Worldline Action

The worldline action for massive spinless compact objects is provided in (1.69). Since we are also interested in the non-relativistic limit, we do not fix the worldline reparametrization gauge, but rather keep the einbein $e(\tau)$ explicit. In this way, it is more convenient to impose $\tau = t$ as the physical time in taking the PN limit. In contrast to the WQFT formalism, since we are only focused on the off-shell potential rather than physical observables, we will only integrate out the intermediate gravitational field but keep the worldline coordinate arbitrary. Expanding the metric in the weak field limit (1.36), and adopting de Donder gauge (1.39), we can extract the Feynman rules from the perturbative Lagrangian. The graviton propagator is given in (1.40), and the worldline coupled to a single graviton is



$$= -i\kappa\ e(\tau)u^\mu(\tau)u^\nu(\tau), \tag{5.1}$$

where we use $u^\mu(\tau)$ to denote the four-velocity. We also need the three-graviton vertex, which is already given in (1.41). For classical effective action, one should be careful about the causality structure of the propagator [146]. Let us consider the Feynman propagator in coordinate space by taking Fourier transform,

$$\begin{aligned}\bar{D}_{ij} &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 + i\varepsilon} e^{ik\cdot x_{ij}} \\ &= \frac{1}{4\pi^2} \frac{i}{x_{ij}^2 - i\varepsilon} = -\frac{1}{4\pi}\delta(x_{ij}^2) + \mathrm{pv}\left(\frac{i}{4\pi^2 x_{ij}^2}\right),\end{aligned} \tag{5.2}$$

where $x_{ij}^\mu = x_i^\mu - x_j^\mu$, and we have used the distributional identity

$$\lim_{\varepsilon\to 0^+} \frac{1}{y \pm i\varepsilon} = \mathrm{pv}\left(\frac{1}{y}\right) \mp i\pi\delta(y) \tag{5.3}$$

in the last step. Since our main concern is the classical, conservative effective action in the post-Minkowskian limit, we only need the classical part of the propagator

$$D_{ij} = \mathrm{Re}(\bar{D}_{ij}) = -\frac{1}{4\pi}\delta(x_{ij}^2). \tag{5.4}$$

We check that it still satisfies the Green's function identity

$$\Box D_{ij} = -\delta^{(4)}(x_{ij}). \tag{5.5}$$

An alternative expression that is easier to take the post-Newtonian limit is

$$\delta(x^2) = \frac{\delta(ct - r)}{2r} + \frac{\delta(ct + r)}{2r}, \tag{5.6}$$

where we have separate the time and spatial component of the 4-vector x^μ, and $r = |\mathbf{x}|$ is the 3-dimensional spatial distance. We clearly see that the real part of Feynman propagator is the average of the retarded and advanced propagator, i.e. the time-symmetric propagator.

5.2 Effective Potential to Second Order Post-Minkowskian

The conservative effective action S_{eff} may then be obtained upon integrating out the graviton fluctuations,

$$e^{iS_{\text{eff}}} = \mathcal{N} \int D[h_{\mu\nu}] e^{i\left(S_{\text{EH}} + S_{\text{gf}} + \sum_i^N S_{\text{pm}}^{(i)}\right)}, \tag{5.7}$$

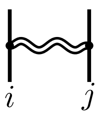
where $\mathcal{N}$ is a normalization constant and we have included N different worldlines. In PM expansion, it can be expressed as

$$S_{\text{eff}} = S^{\text{free}} + \kappa^2 S^{1\text{PM}} + \kappa^4 S^{2\text{PM}} + \mathcal{O}(\kappa^6), \tag{5.8}$$

where the free particle action is

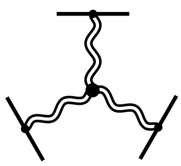
$$S^{\text{free}} = -\sum_i^N \frac{m_i}{2} \int \mathrm{d}\tau_i \, u_i^2(\tau_i). \tag{5.9}$$

Up to 2PM, the N-body potential only includes contributions from two-body and three-body interactions. For simplicity, we will drop the upper bound of the sum N in the following. At 1PM, we only need one diagram, which is a single graviton exchange between two point masses. The full potential at this order includes contributions from any two of the N worldlines. By simply using the Feynman rule (5.1), we obtain

$$\kappa^2 S^{1\text{PM}} = \sum_i \sum_{j \neq i}$$ 

$$= \sum_i \sum_{j \neq i} \int \mathrm{d}\hat{\tau}_i \mathrm{d}\hat{\tau}_j \frac{\kappa^2 m_i m_j}{32\pi} \left[u_{ij}^2 - \frac{1}{2} u_i^2 u_j^2 \right] \delta(x_{ij}^2) \,, \tag{5.10}$$

where we have defined $u_{jk} := u_j \cdot u_k$, and $d\hat{\tau}_i := e_i \mathrm{d}\tau_i$ is reparametrization invariant. At this order, we do not have genuine three-body interactions. The first three-body contribution appears at 2PM, and it arises from a Feynman diagram with a three-graviton vertex. In coordinate space, it reads



$$= \kappa^4 \int \frac{\mathrm{d}^3 \hat{\tau}}{(4\pi)^3} P(x_1(\tau_1), x_2(\tau_2), x_3(\tau_3)), \tag{5.11}$$

where we have introduced the shorthand $\mathrm{d}^3\hat{\tau} = \mathrm{d}\hat{\tau}_1\mathrm{d}\hat{\tau}_2\mathrm{d}\hat{\tau}_3$, as well as

$$\begin{aligned} 8(m_1m_2m_3)^{-1}P(x_i(\tau_i)) :=& \pi\big(4u_{12}^2u_3^2 - 4u_{12}u_{13}u_{23} - u_1^2u_2^2u_3^2\big)\delta(x_{12}^2)\delta(x_{13}^2) \\ &+ \big(u_{12}^2u_3^\mu u_3^\nu - \tfrac{1}{2}u_1^2u_2^2u_3^\mu u_3^\nu + 2u_{13}u_{23}u_2^\mu u_1^\nu\big)\partial_{x_1,\mu}\partial_{x_2,\nu}I_{3\delta} \\ &+ (\text{cyclic}). \end{aligned} \tag{5.12}$$

As we will see below, the integral $I_{3\delta}$ plays a crucial role in obtaining the explicit form of the potential. We leave the detailed analysis for later. Let us first work out the full $S^{2\mathrm{PM}}$. We note that all terms proportional to $u_i \cdot \partial_{x_i}$ have been discarded. They can be written as derivatives with respect to the τ_i parameters $\mathrm{d}/\mathrm{d}\tau_i$. In this case, we can do integration by parts so that the τ_i-derivatives act only on the velocity u_i and the einbein e_j. The former yield terms involving accelerations, which, by proper field redefinition of x_i (or in other words, coordinate transformation), can be pushed to the next order in PM expansion, cf. [147]. For the ones acting on e_i, we can use the equations of motion to replace the einbein, which turns out depending only on the u_j, so again the τ_i-derivatives give terms with accelerations. In the end, both cases are irrelevant at second order in PM, and we will ignore terms involving $u_i \cdot \partial_{x_i}$ throughout this chapter.

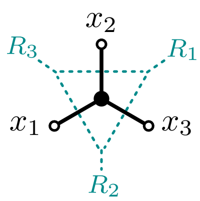
In addition to the genuine three-body interaction (5.11), we also need to include the two-body contributions to complete the N-body action at 2PM. One could in principle draw a two-body diagram with one three-graviton vertex and use Feynman rules to obtain the result, but due to the similarity between this diagram and the three-body one (5.11), there is a simpler and faster way to do so. This is achieved by just identifying two of the three worldlines in (5.11) and then multiplying it with a symmetry factor 1/2. Moreover, propagators that have both ends on the same worldline vanish in dimensional regularization. Therefore, the full 2PM N-body potential thus becomes

$$S^{2\mathrm{PM}} = \frac{1}{6}\int \frac{\mathrm{d}^3\hat{\tau}}{(4\pi)^3} \sum_{i,j,k}{}' P(x_i(\tau_1), x_j(\tau_2), x_k(\tau_3)), \tag{5.13}$$

where the sum $\sum_{i,j,k}'$ runs over $i,j,k = 1,2,3,\ldots,N$ but excludes $i=j=k$. The factor 1/6 is included to remove the over-counting by the sum.

As mentioned above, the integral

$$I_{3\delta} := \int \mathrm{d}^4x_0\,\delta(x_{01}^2)\delta(x_{02}^2)\delta(x_{03}^2) = \tag{5.14}$$



is of central interest in (5.12) for the three-body contribution to the effective potential. In this chapter it arises as the one-loop three-point integral in coordinate space (solid black diagram). Alternatively, it can be understood as the generalized maximal cut of the momentum space triangle integral (green dashed diagram) expressed in terms of region momenta x_j, which maps to the dual momenta R_j via

$$R_j^\mu := x_{j+1}^\mu - x_{j-1}^\mu. \tag{5.15}$$

Moreover, $I_{3\delta}$ may also be obtained from a generalized cut of the four-point (box) integral in the limit where one point is sent to infinity. The box integral is invariant under a Yangian algebra, an extension of its well-known conformal symmetry [148, 149]. As such, in the region $R_j^2 < 0$, the integral is given by the minimal transcendentally solution of the Yangian constraints found in [150] (modulo a piecewise constant):

$$I_{3\delta} = \frac{C}{\sigma}, \qquad \sigma^2 := (R_2 \cdot R_3)^2 - R_2^2 R_3^2. \tag{5.16}$$

Note that due to $R_1 + R_2 + R_3 = 0$ this representation is not unique and one may pick any two R_i's to define σ^2. In regions other than $R_j^2 < 0$, we need a more systematic analysis.

5.3 The 3δ Integral in PM expansion

We will follow Westpfahl's way of evaluating the $I_{3\delta}$ for the retarded propagator [151], and generalize his approach to the case with Feynman propagator. In fact, when $R_j^2 < 0$ for all $j = 1, 2, 3$ the expression (5.16) can be compared with the result of [151] to fix the constant $C(\sigma^2 > 0, R_j^2 < 0) = \pi/4$. However, for generic kinematics, one needs to carefully divide the whole kinematic space into several regions and compute the integral in each of them separately. Interestingly, a detailed calculation shows that the value of the integral depends on the sign of $-\sigma^2$, which may be seen as the square of the area of the parallelogram spanned by x_1, x_2, x_3. It characterizes the space $\mathcal{M}$ spanned by these three points:

1. $\sigma^2 > 0$, $\quad \mathcal{M}$ is 2D Minkowskian,
2. $\sigma^2 = 0$, $\quad \mathcal{M}$ is a 1D straight line,
3. $\sigma^2 < 0$, $\quad \mathcal{M}$ is 2D Euclidean.

The explicit calculation performed below yields

1. $\sigma^2 > 0$, $\quad I_{3\delta} = \frac{\pi}{4\sigma}\Theta(-R_1^2 R_2^2 R_3^2)$,
2. $\sigma^2 = 0$, $\quad I_{3\delta}$ diverges,
3. $\sigma^2 < 0$, $\quad I_{3\delta} \propto \sum_i \delta(R_j^2)$.

Here Θ denotes the Heaviside step function defined as

$$\Theta(x) = \begin{cases} 1, & \text{for} \quad x > 0, \\ \frac{1}{2}, & \text{for} \quad x = 0, \\ 0, & \text{for} \quad x < 0. \end{cases} \tag{5.17}$$

We now explicitly evaluate the $I_{3\delta}$ integral for these three cases, generalizing the computation of Westpfahl [151] for the three-point integral with retarded propagators.

The case $\sigma^2 > 0$. We choose four basis vectors $R_2^\mu, R_3^\mu, \xi_1^\mu, \xi_2^\mu$ such that we can express the integration vector as

$$x_{01}^\mu = \tau R_2^\mu + \bar{\tau} R_3^\mu + r(\cos\varphi \xi_1^\mu + \sin\varphi \xi_2^\mu), \quad r \geq 0. \tag{5.18}$$

Here $\{\xi_1, \xi_2\}$ denotes the (orthogonal) unit basis of the perpendicular complement of $\mathcal{M}$:

$$\xi_i \cdot R_j = 0, \qquad \xi_i \cdot \xi_j = -\delta_{ij}, \quad \text{for } i, j = 1, 2. \tag{5.19}$$

In these coordinates, the integration measure reads

$$\mathrm{d}^4x_0 = \tfrac{1}{2}\sigma \mathrm{d}\tau \mathrm{d}\bar{\tau}\mathrm{d}r^2 \mathrm{d}\varphi, \tag{5.20}$$

and the integral simplifies to

$$\int \frac{\mathrm{d}^4x_0}{4\sigma^2}\delta\big(r^2 + \frac{R_1^2R_2^2R_3^2}{4\sigma^2}\big)\delta\left(\tau - \frac{R_3^2 R_1\cdot R_2}{2\sigma^2}\right)\delta\left(\bar{\tau} + \frac{R_2^2 R_1\cdot R_3}{2\sigma^2}\right). \tag{5.21}$$

This straightforwardly yields

$$I_{3\delta}(\sigma^2 > 0) = \frac{\pi}{4\sigma}\Theta\left(-R_1^2R_2^2R_3^2\right). \tag{5.22}$$

Hence, we conclude that in the region $\sigma^2 > 0$ the piecewise constant in (5.16) is given by

$$C(\sigma^2 > 0) = \frac{\pi}{4}\Theta\left(-R_1^2R_2^2R_3^2\right). \tag{5.23}$$

Note that since $\delta(x_{01}^2)$ is the Green's function of the d'Alembertian, see (5.5), the above integral $I_{3\delta}$ satisfies

$$\partial_1^2 I_{3\delta} = 4\pi\delta(x_{12}^2)\delta(x_{13}^2). \tag{5.24}$$

In the region $\sigma^2 > 0$ this is guaranteed by the Heaviside function in (5.22); dropping the Θ-function in (5.22) would yield a vanishing result as $\partial_1^2\sigma^{-1} = 0$.

The case $\sigma^2 < 0$. In the region where $\sigma^2 < 0$, we can span x_{01}^μ as

$$x_{01}^\mu = tT^\mu + \tau R_2^\mu + \bar{\tau}R_3^\mu + r\xi^\mu. \tag{5.25}$$

Here T^μ and ξ^μ denote again unit vectors that are orthogonal to each other and to R_2^μ, R_3^μ, with T^μ being time-like and ξ^μ space-like. The volume element in this coordinate system is

$$\mathrm{d}^4x_0 = \sqrt{-\sigma^2}\mathrm{d}t\mathrm{d}r\mathrm{d}\tau\mathrm{d}\bar{\tau}, \tag{5.26}$$

and the integral becomes

$$\begin{aligned} I_{3\delta} =& \int \mathrm{d}^4x_0\, \delta\big(t^2 + (\tau R_2^\mu + \bar{\tau}R_3^\mu)^2 - r^2\big)\delta\big(2\tau R_2\cdot R_3 + (2\bar{\tau}+1)R_3^2\big) \\ & \times \delta\big(2\bar{\tau}R_2\cdot R_3 + (2\tau - 1)R_2^2\big) \\ =& \frac{\sqrt{-\sigma^2}}{-4\sigma^2}\int \mathrm{d}t\mathrm{d}r\,\delta\left(t^2 - r^2 - \frac{R_1^2R_2^2R_3^2}{4\sigma^2}\right) \\ =& \frac{1}{4\sqrt{-\sigma^2}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}r}{\sqrt{r^2+1}} \to \infty. \end{aligned} \tag{5.27}$$

Hence, for $\sigma^2 < 0$ the integral diverges.

The case $\sigma^2 = 0$. Finally, for $\sigma^2 = 0$ the surface spanned by the vectors connecting x_1, x_2 and x_3 degenerates into a line. We define the unit vector on this line as R_u^μ, and we set $R_i^\mu = \omega_i R_u^\mu$. Depending on the nature of this line one finds different expressions as follows. For the line being time-like we have

$$\begin{aligned} x_{01}^\mu &= \tau R_u^\mu + r\left(\xi_1^\mu \cos\theta + \xi_2^\mu \sin\theta\cos\phi + \xi_3^\mu \sin\theta\sin\phi\right), \\ \mathrm{d}^4 x_0 &= r^2 \sin\theta \mathrm{d}\tau \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi, \end{aligned} \tag{5.28}$$

and thus

$$\begin{aligned} I_{3\delta} &= \int \mathrm{d}^4 x_0 \delta\left(\tau^2 - r^2\right)\delta\left(\omega_3^2 + 2\tau\omega_3\right)\delta\left(\omega_2^2 - 2\tau\omega_2\right) \\ &= \begin{cases} \infty & \omega_1\omega_2\omega_3 = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.29}$$

For a space-like line and with $T \cdot R_u = 0$, we have

$$\begin{aligned} x_{01}^\mu &= tT^\mu + \tau R_u^\mu + r\left(\xi_1^\mu \cos\theta + \xi_2^\mu \sin\theta\right), \\ \mathrm{d}^4 x_0 &= r \mathrm{d}t \mathrm{d}\tau \mathrm{d}r \mathrm{d}\theta, \end{aligned} \tag{5.30}$$

which implies

$$\begin{aligned} I_{3\delta} &= \int \mathrm{d}^4 x_0 \delta\left(t^2 + \tau^2 - r^2\right)\delta\left(\omega_3^2 + 2\tau\omega_3\right)\delta\left(\omega_2^2 - 2\tau\omega_2\right) \\ &= \begin{cases} \infty & \omega_1\omega_2\omega_3 = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.31}$$

And finally for a light-like line with $T \cdot R_u \neq 0$, we obtain

$$\begin{aligned} x_{01}^\mu &= tT^\mu + \tau R_u^\mu + r\left(\xi_1^\mu \cos\theta + \xi_2^\mu \sin\theta\right), \\ \mathrm{d}^4 x_0 &= \sqrt{(T \cdot R_u)^2 - T^2 R_u^2}\, r \mathrm{d}t \mathrm{d}\tau \mathrm{d}r \mathrm{d}\theta, \end{aligned} \tag{5.32}$$

such that

$$\begin{aligned} I_{3\delta} &= \int \mathrm{d}^4 x_0 \delta\left(t^2 + 2t\tau T \cdot R_u - r^2\right)\delta\left(2t\omega_3 T \cdot R_u\right)\delta\left(2t\omega_2 T \cdot R_u\right) \\ &\sim \int \mathrm{d}\tau \delta(0) = \infty. \end{aligned} \tag{5.33}$$

Hence, the result for $\sigma^2 = 0$ may be summarized as

$$I_{3\delta}(\sigma^2 = 0) \sim \delta(R_1^2) + \delta(R_2^2) + \delta(R_3^2). \tag{5.34}$$

In total we thus conclude that the 3δ-integral can be expressed as

$$I_{3\delta} = \begin{cases} \frac{\pi}{4\sigma}\Theta(-R_1^2 R_2^2 R_3^2), & \sigma^2 > 0, \\ \sim \delta(R_1^2) + \delta(R_2^2) + \delta(R_3^2), & \sigma^2 = 0, \\ \infty, & \sigma^2 < 0. \end{cases} \tag{5.35}$$

We note that when using the result for $\sigma^2 > 0$ it can be useful to expand the theta-function according to

$$\begin{aligned} \Theta\left(-R_1^2 R_2^2 R_3^2\right) = &+ \Theta\left(-R_1^2\right)\Theta\left(-R_2^2\right)\Theta\left(-R_3^2\right) + \Theta\left(-R_1^2\right)\Theta\left(+R_2^2\right)\Theta\left(+R_3^2\right) \\ &+ \Theta\left(+R_1^2\right)\Theta\left(-R_2^2\right)\Theta\left(+R_3^2\right) + \Theta\left(+R_1^2\right)\Theta\left(+R_2^2\right)\Theta\left(-R_3^2\right). \end{aligned} \tag{5.36}$$

5.4 Post-Newtonian Expansion

In this section, we want to provide tests of the above expression for the full 2PM effective action against known results in the PN limit for the three-body potential. For this we first get rid of the einbein by solving the equation of motion $\delta S/\delta e_i = 0$ for e_i perturbatively up to order κ^2:

$$e_i = \frac{1}{\sqrt{u_i^2}} + \sum_{j\neq i}\int \mathrm{d}\tau_j \frac{\kappa^2 m_j}{16\pi\sqrt{u_i^6 u_j^2}}\left(u_{ij}^2 - \frac{1}{2}u_i^2 u_j^2\right) + \mathcal{O}(\kappa^4). \tag{5.37}$$

Plugging this solution back into (5.8) and expanding to order κ^4 yields the 2PM effective action free of the einbein.

In order to do the PN approximation, it is convenient to fix the gauge $\tau_i = t_i =$ coordinate time. As explained in section 1.4, we have

$$u_i^\mu = \left(1, \frac{\mathbf{v}_i}{c}\right), \qquad \kappa \to \frac{\kappa}{c}. \tag{5.38}$$

Additionally, we need to expand the derivative with respect to the coordinate

$$\frac{\partial}{\partial x_i^\mu} = \left(\frac{\partial}{c\partial t_i}, \frac{\partial}{\partial \mathbf{x}_i}\right). \tag{5.39}$$

For convenience, we also rewrite the PN expansion of the propagator for $\delta(x_{ij}^2)$ according to (5.4) and (1.46),

$$\begin{aligned}\delta(x_{ij}^2) =& 4\pi \int \frac{\mathrm{d}^D k}{(2\pi)^D} e^{i\mathbf{k}\cdot\mathbf{r}_{ij}} \sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha \partial_{t_i}^{2\alpha}\delta(t_{ji})}{c^{2\alpha}(\mathbf{k}^2)^{\alpha+1}} \\ =& \frac{\delta(t_i - t_j)}{r_{ij}} - \frac{r_{ij}}{2c^2}\partial_{t_i}\partial_{t_j}\delta(t_i - t_j) + \frac{r_{ij}^3}{24c^4}\partial_{t_i}^2\partial_{t_j}^2\delta(t_i - t_j) + \mathcal{O}(c^{-4}).\end{aligned} \tag{5.40}$$

5.4.1 1PN Expansion

Let us now have a first test of the 2PM potential at order 1PN. Using the PN expansion given in (5.38) and (5.39), we see that in $P(x_i)$ of (5.12) only the first line contributes to leading order in c^{-1}:

$$\sum_{i,j,k}{}' P(x_i) = -\frac{3\pi m_1 m_2 m_3}{8} \sum_i \sum_{\substack{j\neq i \\ k\neq i}} \delta(x_{ij}^2)\delta(x_{ik}^2) + \mathcal{O}(c^{-2}). \tag{5.41}$$

A propagator with both ends on the same worldline is vanishing in dimensional regularization. This allows to rewrite the sum as $\sum'_{i,j,k} \to \sum_i \sum_{j\neq i, k\neq i}$ in the above formula. The non-relativistic expansion of the propagator (5.40) at 1PN localize the time integration. After some rearrangements, we find the 1PN three-body effective action

$$\begin{aligned}S = \sum_i \int dt \Bigg[&-m_i + \frac{1}{c^2}\left(\frac{m_i \mathbf{v}_i^2}{2} + \sum_{j\neq i}\frac{G m_i m_j}{2r_{ij}}\right) + \frac{1}{c^4}\Bigg(\frac{m_i \mathbf{v}_i^4}{8} - \sum_{j\neq i}\sum_{k\neq i}\frac{G^2 m_i m_j m_k}{2 r_{ij} r_{ik}} \\ &+ \sum_{j\neq i}\frac{G m_i m_j}{4 r_{ij}}\Big(6\mathbf{v}_i^2 - (\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) - 7\,\mathbf{v}_i\cdot\mathbf{v}_j\Big)\Bigg)\Bigg],\end{aligned} \tag{5.42}$$

where we denote $\mathbf{n}_{ij} := \mathbf{r}_{ij}/r_{ij}$ as the unit vector pointing from x_j to x_i. This result agrees with the well-known 1PN expression [152].

5.4.2 Integral bootstrap

At order 1PN of the effective potential, only the first line of the three-body contribution (5.12) is needed. Proceeding to order 2PN, we have to take into account the second line as well. This requires to evaluate the PN expansion of the second order derivatives $\partial_j^\mu \partial_k^\nu I_{3\delta}$ of the three-delta integral (5.14). Naively, since we have already calculated $I_{3\delta}$ in the PM limit (5.35), we could simply apply the derivatives to it and then perform the post-Newtonian expansion. One thing we need to clarify is which kinematic region is relevant. As we have seen at 1PN (5.42), the time integration is localized by the τ-delta functions, so we expect the same to happen at 2PN. This means that the three points x_1, x_2, x_3 in the $I_{3\delta}$ integral are on the same time slice of spacetime. Based on (5.16), we can easily see that we are in the region $\sigma^2 > 0$. However, as outlined in detail in appendix B, even though we consider only $\sigma^2 > 0$, the second order derivatives still produce lengthy expressions in terms of Dirac delta functions and their derivatives which are difficult to control. Therefore, we will instead take a different approach which is performing the non-relativistic expansion directly on the level of the integrand of $I_{3\delta}$ and applying the derivatives afterwards. For convenience of the reader we briefly summarize the below strategy: First, we will show that expanding the integrand of $I_{3\delta}$ leads to the family of key integrals given in (5.44). We will then use the Yangian level-one symmetry of these integrals, i.e., invariance under the differential operator (5.46), to obtain the differential equations (5.49). Finally, we explicitly demonstrate how these equations are solved in the form of (5.52), which results in the expressions for the dimensional-regularized integrals that enter into the effective potential.

Using the PN expansion of the propagator (5.40), and expressing it in coordinate space (1.47), we can write the key integral $I_{3\delta}$ in the PN-expansion for general spatial dimension D as

$$I_{3\delta} = \sum_{\alpha,\beta,\gamma=0}^{\infty} \frac{(-1)^{\alpha+\beta+\gamma}}{(2c)^{2(\alpha+\beta+\gamma)}\pi^{3(D/2-1)}} \frac{\Gamma_{\hat\alpha}\Gamma_{\hat\beta}\Gamma_{\hat\gamma}}{\Gamma_{\alpha+1}\Gamma_{\beta+1}\Gamma_{\gamma+1}} \times \int \mathrm{d}t_0 \partial_{t_1}^{2\alpha}\delta(t_{01})\partial_{t_2}^{2\beta}\delta(t_{02})\partial_{t_3}^{2\gamma}\delta(t_{03}) I_3^D[\hat\alpha,\hat\beta,\hat\gamma]. \tag{5.43}$$

Here, $\Gamma_\alpha = \Gamma(\alpha)$ denotes the Gamma-function. We have used the shorthand $\hat\alpha = D/2 - \alpha - 1$ and have introduced the following family of integrals:

$$I_3^D[a_1, a_2, a_3] := \int \frac{\mathrm{d}^D \mathbf{x}_0}{(\mathbf{x}_{01}^2)^{a_1}(\mathbf{x}_{02}^2)^{a_2}(\mathbf{x}_{03}^2)^{a_3}}. \tag{5.44}$$

Note that we are now in pure Euclidean space. These integrals represent the nontrivial central input for the above expansion (5.43), and we will now discuss how to compute them. Notably, in [153] the integrals $I_3^D[a_1, a_2, a_3]$ for generic propagator powers a_j and spacetime dimension D have been expressed in terms of Appell hypergeometric functions F_4, which converges for small values of the effective ratio variables r_{12}/r_{13} and r_{23}/r_{13}. In the present situation, we would like to avoid making assumptions about these ratios, which would imply a limited validity of the resulting effective potential. Moreover, note that here we are merely interested in the special case of half-integer propagator powers a_j in three dimensions, which satisfy the condition

$$a_1 + a_2 + a_3 \leq \frac{D}{2}. \tag{5.45}$$

In particular, this condition implies that the integrals of interest are divergent in strictly three dimensions, and we thus consider their ϵ-expansion around $D = 3$ in dimensional regularization.

Importantly, these integrals are accessible via a bootstrap approach, cf. [150,154]: they feature a non-local Yangian level-one symmetry, i.e., they are annihilated by the differential operator

$$\widehat{\mathrm{P}}^{\mu} := \frac{i}{2}\sum_{k=1}^{3}\sum_{j=1}^{k-1}\Big(\mathrm{P}_j^{\mu}\mathrm{D}_k+\mathrm{P}_{j\nu}\mathrm{L}_k^{\mu\nu}-(j\leftrightarrow k)\Big)+\sum_{j=1}^{3}s_j\mathrm{P}_j^{\mu}, \tag{5.46}$$

where we have used the following representation of the momentum, Lorentz and dilatation generator of the conformal algebra:

$$\begin{aligned}\mathrm{P}_j^{\mu} &= -i\,\partial_{x_j}^{\mu},\\ \mathrm{L}_j^{\mu\nu} &= ix_j^{\mu}\partial_{x_j}^{\nu}-ix_j^{\nu}\partial_{x_j}^{\mu},\\ \mathrm{D}_j &= -ix_{j\mu}\partial_{x_j}^{\mu}-i.\end{aligned} \tag{5.47}$$

The so-called evaluation parameters s_j entering the definition of the level-one generator $\widehat{\mathrm{P}}^{\mu}$ in (5.46) take values [155]

$$\{s_j\} = \frac{1}{2}\{a_2+a_3, a_3-a_1, -a_1-a_2\}. \tag{5.48}$$

Notably, in a dual momentum space, introduced via the transformation (5.15), i.e. $R_j = x_{j+1}-x_{j-1}$, the level-one generator $\widehat{\mathrm{P}}$ translates into a representation of the special conformal generator [154]. Invariance under $\widehat{\mathrm{P}}^{\mu}$ implies two independent partial differential equations (cf. [150] for the PDEs in terms of ratio variables)

$$A_1I_3 = 0, \qquad\qquad A_2I_3 = 0, \tag{5.49}$$

with the second order differential operators

$$\begin{aligned}A_1 =&+r_{12}(\bar{w}_D-2a_2)\partial_{r_{13}}-2r_{12}r_{23}\partial_{r_{13}}\partial_{r_{23}}-r_{12}r_{13}\partial_{r_{13}}^2\\ &+r_{13}(\bar{w}_D+2a_3)\partial_{r_{12}}-2r_{12}^2\partial_{r_{12}}\partial_{r_{13}}-r_{12}r_{13}\partial_{r_{12}}^2,\\ A_2 =&+r_{12}(\bar{w}_D+2a_1)\partial_{r_{23}}-r_{12}r_{23}\partial_{r_{23}}^2-r_{23}(\bar{w}_D+2a_3)\partial_{r_{12}}+r_{12}r_{23}\partial_{r_{12}}^2.\end{aligned} \tag{5.50}$$

Here, for the conformal weight of the integrals (5.44), we have introduced the abbreviation

$$w_D = D-2(a_1+a_2+a_3), \tag{5.51}$$

and $\bar{w}_D = w_D-1$. For $D = 3-2\epsilon$, we make the following ansatz for the ϵ-expansion of the integral I_3, which is inspired by [156]:

$$\mu^{-2\epsilon}I_3^{3-2\epsilon} = \frac{A}{2\epsilon}+B+C\log\left(\frac{r_{12}+r_{13}+r_{23}}{\mu}\right)+\mathcal{O}(\epsilon). \tag{5.52}$$

Here μ denotes some mass scale and A, B, C represent polynomials whose form is constrained by the scaling of the integral:

$$X = \sum_{j=0}^{w_3}\sum_{k=0}^{w_3-j} f_{jk}^{(X)}\, r_{12}^j r_{13}^k r_{23}^{w_3-j-k}. \tag{5.53}$$

For $X \in \{A, B, C\}$ the constant coefficients of the polynomial are denoted by $f_{jk}^{(X)}$. We note that the polynomial B can always be shifted by a term proportional to C via a modification of the mass scale μ. The below results are thus to be understood modulo such a shift. As the coefficients of $1/\epsilon$ and $\log\mu$ are correlated in the ϵ-expansion of (5.52), we must have $A = -C$, which we also find from the bootstrap arguments.

The solution of the homogeneous differential equations (5.49) will depend on some undetermined constants. In general, these can, for instance, be fixed by comparing a coincident point limit of the solution with the following well-known expression for the two-point integral, cf. e.g. [157]:

$$\int \frac{\mathrm{d}^D x_0}{x_{01}^{2a_1} x_{02}^{2a_2}} = \pi^{\frac{D}{2}} \frac{\Gamma_{a_1+a_2-\frac{D}{2}} \Gamma_{\frac{D}{2}-a_1} \Gamma_{\frac{D}{2}-a_2}}{\Gamma_{a_1}\Gamma_{a_2}\Gamma_{D-a_1-a_2}} r_{12}^{D-2a_1-2a_2}. \tag{5.54}$$

However, for the lower propagator powers considered below, some of the arguments of the Gamma-functions will actually be zero. It is thus useful to note that the Laplacian acting on leg 1 of the integral generates a recursive structure on the above integrals, e.g.

$$\Delta_1 I_3[a_1, a_2, a_3] = 2a_1(2a_1 + 2 - D) I_3[a_1 + 1, a_2, a_3], \tag{5.55}$$

and similar for legs 2 and 3. This equation can alternatively be used to relate the undetermined coefficients for integrals with negative propagator powers to the leading-order 'seed' integral $I_3\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$.

In the following, we bootstrap the integrals contributing to the leading terms of the non-relativistic expansion (5.43) using the level-one Yangian PDEs (5.50). We have compared the expansion of the below results for small ratios r_{12}/r_{13} and r_{23}/r_{13} to the expressions in terms of Appell hypergeometric functions given in [153] finding full agreement, see also [154] for our conventions. The following integrals serve as input for the three-body effective potential via (5.43) and (5.12).

Order c^0: At leading order, only one integral $I_3[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ contributes to the expansion (5.43). The finite part of the ϵ-expansion around $D = 3$ is already known since 1970s by Ohta et. al. [156]. Here we evaluate the integral by solving the PDEs (5.50) with the ansatz (5.52), which yields

$$\mu^{-2\epsilon} I_3\left[\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right] = \frac{b_1}{2\epsilon} - b_2 \log\left(\frac{r_{12} + r_{13} + r_{23}}{\mu}\right) + \mathcal{O}(\epsilon), \tag{5.56}$$

where b_1,b_2 are integral constants. They can be fixed by taking the coincidence two-point limit where $x_2 = x_1$ and comparing it with (5.54),

$$b_1 = b_2 = 4\pi. \tag{5.57}$$

Note that we neglect a additional finite constant that can be shifted by changing the mass scale μ.

Order c^{-2}: Analogous to the above, at next-to-leading order we need to evaluate $I_3[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$. The ϵ-expansion of the integral is again obtained by solving the PDEs (5.50),

$$\begin{aligned} \mu^{-2\epsilon} I_3\left[\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right] = &- c\frac{(r_{12}^2 - r_{13}^2 - r_{23}^2)}{2\epsilon} + c\Big[(r_{12} - r_{13})(r_{12} - r_{23}) \\ &+ (r_{12}^2 - r_{13}^2 - r_{23}^2)\log\left(\frac{r_{12} + r_{13} + r_{23}}{\mu}\right)\Big] + \mathcal{O}(\epsilon). \end{aligned} \tag{5.58}$$

Here we can verify the relation $A = -C$ in the ansatz (5.52). Again, the finite part is defined modulo the polynomial in the divergent $1/\epsilon$ contribution. The constant c can be fixed by comparing with the two-point function as before. Alternatively, we can also fix it by applying the Laplacian on leg 3 to the integral

$$\Delta_3 I_3\left[\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right] = \frac{6c}{\epsilon} - 4c - 12c\log\left(\frac{r_{12}+r_{13}+r_{23}}{\mu}\right) \tag{5.59}$$

and using the recursion relation (5.55),

$$\Delta_3 I_3\left[\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right] = (2-2\epsilon) I_3\left[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right]. \tag{5.60}$$

Note that due to the uncertainty of the finite part, the equation holds only up to a shift by a constant. Comparing the right-hand sides of (5.59) and (5.60), we fine the relation between c and the constant b_1, b_2,

$$c = \frac{b_1}{6} = \frac{b_2}{6} = \frac{2\pi}{3}. \tag{5.61}$$

The result (5.58) will be employed to obtain the novel contributions to the 3PN three-body effective potential, which scale as $\frac{G^2 m^3 v^4}{c^8 r^2}$, see subsection 5.4.4.

Order c^{-4}: We would like to demonstrate that the above bootstrap approach can be easily applied to higher orders. For that, let us consider order c^{-4} of the non-relativistic expansion (5.43). We see that two integrals $I_3[-\frac{1}{2},-\frac{1}{2},\frac{1}{2}]$ and $I_3[\frac{1}{2},\frac{1}{2},-\frac{3}{2}]$ are relevant. However, the contributions to the 4PN effective potential which need these integrals are too lengthy to fit in this thesis, so we will not evaluate them. Solving the partial differential equations (5.50) with the ansatz (5.52), we find that the polynomials read

$$A\left[-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\right] = d_1\left(-\frac{1}{3}r_{12}^4 + (r_{13}^2 - r_{23}^2)^2 - \frac{2}{3}r_{12}^2(r_{13}^2 + r_{23}^2)\right), \tag{5.62}$$

$$\begin{aligned} B[-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}] =& d_1\Bigg(r_{12}\left(r_{13}-r_{23}\right)^2\left(r_{13}+r_{23}\right) - \frac{r_{12}^4}{9} - \frac{1}{3}r_{12}^3\left(r_{13}+r_{23}\right) \\ &+ r_{13}r_{23}\left(r_{13}-r_{23}\right)^2 - \frac{1}{9}r_{12}^2\left(5r_{13}^2 - 3r_{13}r_{23} + 5r_{23}^2\right)\Bigg), \end{aligned} \tag{5.63}$$

as well as

$$A[\tfrac{1}{2},\tfrac{1}{2},-\tfrac{3}{2}] = d_2\Big(2r_{12}^2\left(r_{13}^2+r_{23}^2\right) - r_{12}^4 - r_{13}^4 - r_{23}^4 - \tfrac{2}{3}r_{13}^2 r_{23}^2\Big), \tag{5.64}$$

$$\begin{aligned} B[\tfrac{1}{2},\tfrac{1}{2},-\tfrac{3}{2}] =& d_2\Big(\frac{4r_{12}^4}{3} - r_{13}r_{23}^3 + \frac{4}{9}r_{13}^2 r_{23}^2 - r_{13}^3 r_{23} - r_{12}^3\left(r_{13}+r_{23}\right) \\ &- r_{12}^2\left(2r_{13}^2 - r_{23}r_{13} + 2r_{23}^2\right) + \frac{r_{12}}{3}\left(5r_{13}^3 + 3r_{23}r_{13}^2 + 3r_{23}^2 r_{13} + 5r_{23}^3\right)\Big). \end{aligned} \tag{5.65}$$

Again, the overall constants d_1, d_2 are fixed by matching to the coefficients for the integral (5.58) via the recursion (5.55), which yields,

$$d_1 = -\frac{\pi}{10}, \qquad d_2 = -\frac{3\pi}{10}. \tag{5.66}$$

Note again that the polynomials $B[-\frac{1}{2},-\frac{1}{2},\frac{1}{2}], B[\frac{1}{2},\frac{1}{2},-\frac{3}{2}]$ are only defined up to a shift by the corresponding A due to the arbitrariness of the mass scale μ. This is sufficient to generate the 4PN order G^2 contributions to the three-body potential.

5.4.3 2PN Expansion and the Two Body Limit

With the integral (5.56), we can now move on to compute order 2PN of the effective potential following the same way as the 1PN order in 5.4.1. Since the potential in (5.13) includes both the two-body and the genuine three-body interactions, it is convenient to treat them separately. For this purpose, let us decompose the sum as

$$\sum_{i,j,k}{}' \to \sum_i \sum_{j\neq i} \sum_{k\neq i,j} + \Big(\sum_i \sum_j \Big|_{k=i} + \text{(cyclic)} \Big), \tag{5.67}$$

where the first term on the right-hand side accounts for the three-body interaction and the remaining terms are the two-body interactions. We will compute the summands for different indices $i \neq j \neq k$. The two-body case can be obtained by simply identifying two of the three indices. However, there are two difficulties in this process. The first one is that some denominators will be vanishing in this limit, such as $1/r_{ij}|_{j=i}$, which will lead to divergence expressions. To tackle this problem, we propose to regularize the divergences as

$$\frac{1}{r_{ij}}\Big|_{j=i} \to 0. \tag{5.68}$$

This is consistent with the fact that propagators with both ends on the same worldline should be neglected in dimensional regularization. The other difficulty is that the unit vector $\mathbf{n}_{ij}|_{j=i}$ is indefinite in the two-body limit. To solve this, we develop the following mapping from three-body to two-body interactions. Firstly, terms of odd order in $\mathbf{n}_{ij}|_{j=i}$ are vanishing due to the anti-symmetry in the indices. Secondly, for terms quadratic in $\mathbf{n}_{ij}|_{j=i}$, we adopt the prescription:

$$\mathbf{n}_{ij}\cdot\mathbf{v}_\alpha\, \mathbf{n}_{ij}\cdot\mathbf{v}_\beta\big|_{j=i} \to \mathbf{v}_\alpha \cdot \mathbf{v}_\beta. \tag{5.69}$$

This means that practically, if the identification of indices yields a term that is quadratic in $\mathbf{n}_{ij}|_{j=i}$ and includes a factor on the left-hand side of (5.69), we replace this factor by the right-hand side. This mapping is justified by the perspective of dimensional analysis and symmetry considerations, and more importantly, it reproduces the correct potential at 2PN as given in the literature. Further more, at 3PN order that will be discussed in the next subsection 5.4.4, we will encounter terms that are quartic in $\mathbf{n}_{ij}|_{j=i}$. For that, we also present the mapping rule here

$$\mathbf{n}_{ij}\cdot\mathbf{v}_\alpha\, \mathbf{n}_{ij}\cdot\mathbf{v}_\beta\, \mathbf{n}_{ij}\cdot\mathbf{v}_\rho\, \mathbf{n}_{ij}\cdot\mathbf{v}_\sigma\big|_{j=i} \;\to\; \mathbf{v}_\alpha\cdot\mathbf{v}_\beta\, \mathbf{v}_\rho\cdot\mathbf{v}_\sigma + \mathbf{v}_\alpha\cdot\mathbf{v}_\rho\, \mathbf{v}_\sigma\cdot\mathbf{v}_\beta + \mathbf{v}_\alpha\cdot\mathbf{v}_\sigma\, \mathbf{v}_\beta\cdot\mathbf{v}_\rho. \tag{5.70}$$

To obtain the effective potential, we need apply the double derivative to the integral (5.4.4). We note that the ϵ divergences naturally drop out as expected. This property is conjectured to be true to all orders in the PN expansion. At higher orders, the polynomial of the divergent part will be of higher degrees, but we also have more derivatives in the PN expansion (5.43) to cancel the $1/\epsilon$ term. We explicitly verify it in the 3PN calculation of 5.4.4. From (5.12), we can easily compute the 2PN effective action,

$$S^{\rm 2PN} = \sum_i \int \frac{{\rm d}t}{c^6}\Bigg\{\frac{m_i \mathbf{v}_i^6}{16} + \sum_{j\neq i}\frac{G m_i m_j}{16 r_{ij}}\Big[3(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 - 6\,\mathbf{n}_{ij}\cdot\mathbf{v}_i\,\mathbf{n}_{ij}\cdot\mathbf{v}_j\,\mathbf{v}_{ij}^2 - 2\,(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\,\mathbf{v}_i^2$$
$$+3\,\mathbf{v}_i^2\,\mathbf{v}_j^2 + 2\,(\mathbf{v}_i\cdot\mathbf{v}_j)^2 - 20\,\mathbf{v}_i^2\,\mathbf{v}_i\cdot\mathbf{v}_j + 14\,\mathbf{v}_i^4\Big] + \sum_{j\neq i}\frac{G^2 m_i m_j^2}{2 r_{ij}^2}\Big[33\,(\mathbf{n}_{ij}\cdot\mathbf{v}_{ij})^2 - 17\mathbf{v}_{ij}^2\Big]$$
$$+\sum_{j\neq i}\sum_{k\neq i}\frac{G^2 m_i m_j m_k}{8}\Bigg[\frac{1}{r_{ij}r_{ik}}\big(4(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 + 18\,\mathbf{v}_i^2 - 16\,\mathbf{v}_j^2 - 32\,\mathbf{v}_i\cdot\mathbf{v}_j + 32\,\mathbf{v}_j\cdot\mathbf{v}_k\big)$$
$$+\frac{1}{r_{ij}^2}\big(14\,\mathbf{n}_{ik}\cdot\mathbf{v}_k\,\mathbf{n}_{ij}\cdot\mathbf{v}_k - 12\,\mathbf{n}_{ij}\cdot\mathbf{v}_i\,\mathbf{n}_{ik}\cdot\mathbf{v}_k + \mathbf{n}_{ij}\cdot\mathbf{n}_{ik}\,(\mathbf{n}_{ik}\cdot\mathbf{v}_k)^2 - \mathbf{n}_{ij}\cdot\mathbf{n}_{ik}\,\mathbf{v}_k^2\big)\Bigg]$$
$$+\sum_{j\neq i}\sum_{k\neq i,j} G^2 m_i m_j m_k\Bigg[\frac{2(\mathbf{n}_{ij}-\mathbf{n}_{jk})\cdot\mathbf{v}_{ij}}{(r_{ij}+r_{ik}+r_{jk})^2}\big(4\,(\mathbf{n}_{ij}+\mathbf{n}_{ik})\cdot\mathbf{v}_{ij} + (\mathbf{n}_{ik}+\mathbf{n}_{jk})\cdot\mathbf{v}_{ik}\big)$$
$$+\frac{9\,(\mathbf{n}_{ij}\cdot\mathbf{v}_{ij})^2 - 9\,\mathbf{v}_{ij}^2 + 2\,(\mathbf{n}_{ij}\cdot\mathbf{v}_{ik})^2 - 2\,\mathbf{v}_{ik}^2}{r_{ij}\,(r_{ij}+r_{ik}+r_{jk})}\Bigg]\Bigg\} + G^3\times[\text{static term}]\,, \tag{5.71}$$

where we have defined $\mathbf{v}_{ij} := \mathbf{v}_i - \mathbf{v}_j$. Note that we have implicitly pushed terms that involve accelerations to higher orders in PM expansion via a field redefinition of $\mathbf{x}$. Up to the static term proportional to G^3, which we do not have accessed to since it is in the next order in PM, we have checked that our result against the literature [142, 158]. And we find full agreement after adding a total derivative term. This also serves as a test for the integral $I_3\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]$ (5.56) that we compute from solving the level-one Yangian PDEs.

5.4.4 Contribution to 3PN

In this subsection, we compute the novel contribution to the 3PN N-body effective potential from the 2PM order, which scales as G^2v^4 and includes both the two-body and genuine three-body terms. In addition to that, we also have the G^1v^6 terms that stems from the 1PM potential (5.10). Due to the complexity of the final result, the full 3PN action can be formally expressed as

$$S^{\rm 3PN} = \sum_i \int \frac{{\rm d}t}{c^8}\left\{\frac{5}{128}m_i\mathbf{v}_i^8 + L^{\rm 3PN}_{(A)} + L^{\rm 3PN}_{(B)} + L^{\rm 3PN}_{(C)} + L^{\rm 3PN}_{(D)}\right\} + \mathcal{O}(G^3). \tag{5.72}$$

Analogous to the 2PN potential (5.71), here we do not have the terms at order G^3, which require two yet unknown four-point integrals at one and two loops. Further more, to have the complete 3PN N-body effective action, the G^4 contributions should also be included, which are out of scope of this section. For the terms from 1PM and 2PM, we have classified them in (5.72) based on the power of G and the structure of summations. The terms linear in G are from the PN expansion of the 1PM action and is collected in $L^{\rm 3PN}_{(A)}$, which explicitly reads

$$L^{\rm 3PN}_{(A)} = \sum_{j\neq i}\frac{G m_i m_j}{32 r_{ij}}\Big[3\,\mathbf{n}_{ij}\cdot\mathbf{v}_i\,(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\,\big(2\,\mathbf{v}_i^2\mathbf{n}_{ij}\cdot\mathbf{v}_j + 6\,\mathbf{v}_i^2\,\mathbf{n}_{ij}\cdot\mathbf{v}_i - 5\,\mathbf{v}_i\cdot\mathbf{v}_j\mathbf{n}_{ij}\cdot\mathbf{v}_i\big) \tag{5.73}$$
$$+\mathbf{n}_{ij}\cdot\mathbf{v}_i\,\mathbf{n}_{ij}\cdot\mathbf{v}_j\,\big(10\,(\mathbf{v}_i\cdot\mathbf{v}_j)^2 + 8\,\mathbf{v}_i^2\,\mathbf{v}_i\cdot\mathbf{v}_j - 5\,\mathbf{v}_i^2\,\mathbf{v}_j^2 - 14\,\mathbf{v}_i^4\big)$$
$$+2\,(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\,\mathbf{v}_i^2\,\big(5\mathbf{v}_i\cdot\mathbf{v}_j - 3\,\mathbf{v}_i^2\big) - 6\,\mathbf{v}_i^2\,\mathbf{v}_j^2\,(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 + 16\,\mathbf{v}_i^4\,\mathbf{v}_j^2 + 2\,(\mathbf{v}_i\cdot\mathbf{v}_j)^3$$
$$+12\,\mathbf{v}_i^2\,(\mathbf{v}_i\cdot\mathbf{v}_j)^2 - 19\,\mathbf{v}_i^2\,\mathbf{v}_j^2\,\mathbf{v}_i\cdot\mathbf{v}_j - 34\,\mathbf{v}_i^4\,\mathbf{v}_i\cdot\mathbf{v}_j + 22\,\mathbf{v}_i^6 - 5\,(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^3\,(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^3\Big].$$

As before, we added a total derivative as given in (C.1) in appendix C to reduce the expression to the above form, and implicitly a field redefinition is adopted to push accelerations to next order in G. For terms quadratic in G, we separate them to $L^{\rm 3PN}_{(B)}$, $L^{\rm 3PN}_{(C)}$ and $L^{\rm 3PN}_{(D)}$ according to the structure of summations. The two-body interactions of the second line of (5.12) gives rise to $L^{\rm 3PN}_{(B)}$, which reads

$$
\begin{aligned}
L^{\rm 3PN}_{(B)} = \sum_{j\neq i} \frac{G^2 m_i m_j^2}{4 r_{ij}^2} \Big[& \big(-200\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j + 167\,\mathbf{v}_i^2 + 66\,\mathbf{v}_j^2\big)\,(\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_i)^2 - 98\,(\mathbf{v}_i\!\cdot\!\mathbf{v}_j)^2 \\
& - 2\,\big(99\,\mathbf{v}_i^2 + 64\,\mathbf{v}_j^2 - 130\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j\big)\,\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j\,\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_i + 96\,\mathbf{v}_j^2 + 96\,\mathbf{v}_j^2\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j \\
& - 44\,(\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_i - \mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j)^2\,\big(2\,(\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_i)^2 + (\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j)^2\big) + \mathbf{v}_i^2\,\big(134\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j - 49\,\mathbf{v}_j^2\big) \\
& + \big(65\,\mathbf{v}_i^2 - 128\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j\big)\,(\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j)^2 - 51\,\mathbf{v}_i^4 - 32\,\mathbf{v}_j^4 \Big].
\end{aligned} \tag{5.74}
$$

Furthermore, we collect all the contributions from the first line of (5.12) as well as from field redefinitions and total derivatives that are used to remove accelerations in $L^{\rm 3PN}_{(C)}$. Note that the contribution from the first line of (5.12) can be explicitly written in the form before taking derivatives

$$
\begin{aligned}
L^{\rm 3PN}_{(\text{first line})} = \sum_{j\neq i}\sum_{k\neq i} G^2 m_i m_j m_k \Big\{ & \frac{1}{8\,r_{ij} r_{ik}} \Big[2\,\mathbf{v}_j^2 \big(32\,\mathbf{v}_i\!\cdot\!\mathbf{v}_k + 16\,\mathbf{v}_j\!\cdot\!\mathbf{v}_k - 7\,\mathbf{v}_j^2 - 9\,\mathbf{v}_k^2\big) \\
& + \mathbf{v}_i^2\,\big(\mathbf{v}_i^2 - 20\,\mathbf{v}_j^2 + 16\,\mathbf{v}_j\!\cdot\!\mathbf{v}_k\big) + 32\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j\,(\mathbf{v}_i\!\cdot\!\mathbf{v}_j - \mathbf{v}_i\!\cdot\!\mathbf{v}_k - 2\,\mathbf{v}_j\!\cdot\!\mathbf{v}_k) \Big] \\
& + \big(\mathbf{v}_i^2 - 3\,\mathbf{v}_j^2 - 3\,\mathbf{v}_k^2 + 8\,\mathbf{v}_j\!\cdot\!\mathbf{v}_k\big)\,(\mathbf{v}_k\!\cdot\!\partial_{x_k})^2 \frac{r_{ik}}{2\,r_{ij}} \\
& - (\mathbf{v}_k\!\cdot\!\partial_{x_k})^2 (\mathbf{v}_j\!\cdot\!\partial_{x_j})^2 \frac{r_{ij} r_{ik}}{4} - (\mathbf{v}_k\!\cdot\!\partial_{x_k})^4 \frac{r_{ik}^3}{12\,r_{ij}} \Big\}.
\end{aligned} \tag{5.75}
$$

After evaluating the derivatives, $L^{\rm 3PN}_{(C)}$ is expressed as

$$
\begin{aligned}
L^{\rm 3PN}_{(C)} = \sum_{j\neq i}\sum_{k\neq i} \frac{G^2 m_i m_j m_k}{16} \Big\{ & \frac{1}{r_{ij} r_{ik}} \Big[2\,(\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j)^2\,\big(16\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j - 18\,\mathbf{v}_i^2 - 32\,\mathbf{v}_j\!\cdot\!\mathbf{v}_k + 12\,\mathbf{v}_j^2 - (\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_k)^2\big) \\
& + 64\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j\,\big(2\,(\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_k)^2 + \mathbf{v}_i\!\cdot\!\mathbf{v}_k - 2\,\mathbf{v}_j\!\cdot\!\mathbf{v}_k - \mathbf{v}_j^2\big) + 16\,\mathbf{v}_j^2\,\big(8\,\mathbf{v}_j\!\cdot\!\mathbf{v}_k - 2\,\mathbf{v}_j^2 - 2\,\mathbf{v}_k^2 - (\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_k)^2\big) \\
& + 16\,\mathbf{v}_i^2\,\big(3\,\mathbf{v}_j^2 + 2\,\mathbf{v}_j\!\cdot\!\mathbf{v}_k - 10\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j\big) - 6\,(\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j)^4 + 96\,(\mathbf{v}_i\!\cdot\!\mathbf{v}_j)^2 + 49\,\mathbf{v}_i^4 \Big] \\
& + \frac{1}{3\,r_{ij}^2} \Big[20\,(\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_k)^3\,(\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_i - \mathbf{n}_{ij}\!\cdot\!\mathbf{v}_k) - 3\,\mathbf{n}_{ij}\!\cdot\!\mathbf{n}_{ik}\,\big((\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_k)^2 - \mathbf{v}_k^2\big)\,\big(3\,(\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j)^2 + 8\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j \\
& + (\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_k)^2 + 2\,\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_i\,\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_k + 6\,\mathbf{v}_i\!\cdot\!\mathbf{v}_k - 5\,\mathbf{v}_i^2 - 4\,\mathbf{v}_j^2 - 4\,\mathbf{v}_k^2\big) \\
& + 6\,\big((\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_k)^2 - \mathbf{v}_k^2\big)\,\big(3\,\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j\,\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_i - 3\,\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j\,\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_j - 3\,\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_i\,\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_k + 4\,\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_i\,\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_j \\
& - \mathbf{n}_{ij}\!\cdot\!\mathbf{v}_i\,\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_i\big) + 6\,\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_k \big[\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_k\,\big(19\,\mathbf{v}_i^2 + 28\,\mathbf{v}_j^2 - 56\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j - 2\,\mathbf{v}_i\!\cdot\!\mathbf{v}_k - 21\,(\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j)^2\big) \\
& + 2\,\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_i\,\big(11\,\mathbf{v}_i^2 - 12\,\mathbf{v}_j^2 - 23\,\mathbf{v}_i\!\cdot\!\mathbf{v}_k + 28\,\mathbf{v}_j\!\cdot\!\mathbf{v}_k + 9\,(\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j)^2\big) \\
& + 6\,\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_j\,\big(6\,\mathbf{v}_i\!\cdot\!\mathbf{v}_j + 7\,\mathbf{v}_i\!\cdot\!\mathbf{v}_k - 6\,\mathbf{v}_i^2 - 7\,\mathbf{v}_j\!\cdot\!\mathbf{v}_k\big) \big] \\
& + 18\,\mathbf{v}_k^2\,\mathbf{n}_{ik}\!\cdot\!\mathbf{v}_k\,(5\,\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_k - 4\,\mathbf{n}_{ij}\!\cdot\!\mathbf{v}_i) \Big] \Big\}.
\end{aligned} \tag{5.76}
$$

The term $L^{\rm 3PN}_{(D)}$ contributing to (5.72) stems from the genuine three-body interactions from the second line of (5.12). Formally, it can be written as derivative operators acting on the integrals $I_3[\frac{1}{2},\frac{1}{2},\frac{1}{2}]$ and $I_3[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]$,

$$
\begin{aligned}
L^{\rm 3PN}_{(D)} = \sum_{j\neq i}\sum_{k\neq i,j} \frac{G^2 m_i m_j m_k}{4\pi}\Big\{ \Big[&(6\mathbf{v}_i^2+\mathbf{v}_k^2-8\mathbf{v}_i\cdot\mathbf{v}_j)(\mathbf{v}_{ki}\cdot\partial_{x_i})(\mathbf{v}_{kj}\cdot\partial_{x_j}) \\
&+(8\mathbf{v}_{ik}^2-4\mathbf{v}_k^2)(\mathbf{v}_{ji}\cdot\partial_{x_i})(\mathbf{v}_{ij}\cdot\partial_{x_j})\Big] I_3[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}] \\
&+(\mathbf{v}_k\cdot\partial_{x_k})^2\Big[(\mathbf{v}_{ki}\cdot\partial_{x_i})(\mathbf{v}_{kj}\cdot\partial_{x_j})+2(\mathbf{v}_{ik}\cdot\partial_{x_k})(\mathbf{v}_{ij}\cdot\partial_{x_j}) \\
&+4(\mathbf{v}_{ji}\cdot\partial_{x_i})(\mathbf{v}_{ij}\cdot\partial_{x_j})+8(\mathbf{v}_{jk}\cdot\partial_{x_k})(\mathbf{v}_{kj}\cdot\partial_{x_j})\Big] I_3[\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}]\Big\}.
\end{aligned} \tag{5.77}
$$

To take into account the permutation of particles, we denote the external points as i,j,k, so the integrals I_3 depends on them rather than the labels $1,2,3$ as in subsection 5.4.2. The expressions of the integrals $I_3[\frac{1}{2},\frac{1}{2},\frac{1}{2}]$ and $I_3[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]$ as given in (5.56) and (5.58), respectively. For convenience we display them here again:

$$
\mu^{-2\epsilon} I_3[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}] = \frac{2\pi}{\epsilon} - 4\pi\log\left(\frac{r_{ij}+r_{ik}+r_{jk}}{\mu}\right) + \mathcal{O}(\epsilon), \tag{5.78}
$$

$$
\begin{aligned}
\mu^{-2\epsilon} I_3[\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}] = -\frac{2\pi}{3}\Big[&\frac{r_{ij}^2-r_{ik}^2-r_{jk}^2}{2\epsilon} - (r_{ij}-r_{ik})(r_{ij}-r_{jk}) \\
&-(r_{ij}^2-r_{ik}^2-r_{jk}^2)\log\left(\frac{r_{ij}+r_{ik}+r_{jk}}{\mu}\right)\Big] + \mathcal{O}(\epsilon).
\end{aligned} \tag{5.79}
$$

By taking the coincidence limit where two of the three bodies are identified, we can reproduce the two-body contributions (5.74) with the prescriptions presented in (5.68), (5.69) and (5.70). The result is independent on the dimensional regularization as the $1/\epsilon$-poles and the mass scale μ drop out after taking the derivatives in (5.77). This is necessary for physical quantities and serves as a test of our calculation. We check that it is true at least up to 4PN order. Due to the complexity, we present the expression for $L^{\rm 3PN}_{(D)}$ after taking the derivatives in appendix C.

We thus have all the contributions of order G^2 given in (5.72) to the 3PN effective potential. In principal, we could follow the same procedure to compute contributions that scale as $G^2 v^{2n}$ to higher PN order from 2PM. The two integrals we need at next order of the expansion are already given in subsection 5.4.2. However, as we have seen in this subsection, the final expression at 3PN, especially the genuine three-body contributions (C), is already very complicated. Therefore, we refrain from explicitly calculating higher-order contributions of the PN expansion.

5.5 Comments on the three-body problem

The integrals $I_3^D[a_1,a_2,a_3]$ (5.44) with half-integer propagator powers a_1,a_2,a_3 are critical in the computation of three-body PN potentials. There are various way to evaluate them. For example, in this chapter, we show a bootstrap approach based on the Yangian level-one symmetry in subsection 5.4.2. Another way is the Mellin-Barnes methods, which yield a result expressed in terms of Appell hypergeometric series F_4 that is valid for generic propagator powers [153]. However, this formula is inapplicable to the case we consider in this chapter because the F_4 functions converge only if $r_{12}+r_{13}<r_{23}$, which is outside of the physical

region in Euclidean space. Alternatively, one can use the so-called method of regions, in which the integrals are computed as a series expansion in the limit where two of the three bodies are close to each other [145]. Given its important role in obtaining the PN potential, it would be great to have a generic closed form for I_3^D with arbitrary half-integer propagator powers. Yet, to the best of our knowledge, no such formula is known in the literature. We leave that for future researches.

For N-body potential at higher order in PM, we will need similar Feynman integrals at higher loops. For example, the two-loop integrals with a internal massless propagator will show up at 3PM

$$\int \frac{\mathrm{d}^D \mathbf{x}_i \mathrm{d}^D \mathbf{x}_j}{(\mathbf{x}_{i1}^2)^{a_1}(\mathbf{x}_{i2}^2)^{a_2}(\mathbf{x}_{ij}^2)^{b}(\mathbf{x}_{j3}^2)^{a_3}(\mathbf{x}_{j4}^2)^{a_4}}, \tag{5.80}$$

where a_1, a_2, a_3, a_4, b are positive half integers. Computation of these integrals is the main obstacle in obtaining higher order PM and PN results. The leading contribution with $a_1 = a_2 = a_3 = a_4 = b = 1/2$ gives the static terms in (5.71) which completes the 2PN effective potential. Generically, at nPM we need a family of $(n-1)$-loop integrals with $n-2$ internal propagators. Evaluating these integrals is challenging, but modern Feynman integral techniques such as the ones we introduce for the three-body 1-loop ones could provide novel perspective to attack the problem.

Chapter 6

Summary and Outlook

The color-kinematics duality and the double copy relation present a novel insight on connections of various classes of theories, revealing aspects that are obscure by looking at the Lagrangian or the Hamiltonian. They are well-studied for scattering amplitudes and have been used to translate results between gauge and gravity theories. It is well-known that the duality relation applies to many other classes of theories and beyond scattering amplitudes. Therefore, it is promising to consider some extensions of the pure YM/gravity double copy that can be applied to high precision prediction in gravitational physics.

In this thesis, we study several approaches of incorporating massive particles into the gauge/gravity story, which model the dynamics of heavy (charged) astronomical objects such as black holes in the corresponding backgrounds. Chapter 2 examines the quantum aspects by considering the double copy of tree-level scattering amplitudes of massive scalar fields coupled to Yang-Mills theory. We conclude that the emerging theory accounts for heavy spinless particles coupled to the $\mathcal{N}=0$ supergravity in a minimal way, with additional short-range self-interactions. In chapter 3, we move to the pure classical setup by using worldlines to describe massive matters. We propose a double copy prescription for the eikonal phase in bi-adjoint, YM, and dilaton-gravity background, and check it explicitly up to next-to-leading order in the coupling constants. Chapter 4 considers the probe limit of the two-body problem, which has the privilege of being able to solve the equations of motion non-perturbatively. As an extension of the classical Kerr-Schild double copy, a mapping relation between the conserved charges in YM and gravity background is discovered (4.9). Aside from the double copy relation, in chapter 5, we also obtain the three-body potential in general relativity directly from an effective worldline formalism. We formally present the 2PM integrand and provide a way to extract PN contributions by explicitly calculating the G^2v^4 terms.

Based on these projects, there are many interesting directions to explore. For example, we could incorporate spin effects into many of the aforementioned projects, so that they can be applied to some astronomical objects such as Kerr black holes or neutron stars. In perturbative double copy in chapter 2 and 3, it is desirable to have a systematic way to remove the dilaton and the two-form in the resulting gravitational theory in order to describe real-world physics. Moreover, the double copy of the conserved quantities of geodesic motions is built upon the non-perturbative Kerr-Schild relation, which is distinct from the scattering amplitudes story. Understanding the how these seemly different methods related to each other will deepen our understanding on the duality between color and kinematics. Besides, we would also like to push our calculation of the N-body effective potential to higher orders in PM and PN expansions.

Surprisingly, the color-kinematics duality extends to theories that have no obvious connection to gauge or gravity theories. They are connected by the double copy relations and form a "web of theories" [39, 40]. This striking fact implies that there are nontrivial constraints shared by these consistent theories. Understanding these properties may help to unravel the true nature of the double copy.

Many attempts have been made to generalize the notion of double copy beyond quantum scattering amplitudes, but it is not yet clear how far this extension can reach. In 4-dimensional Einstein gravity, classical solutions that display double copy structures have been extended to Petrov type D. However, since metrics are dependent on the choice of gauge, an appropriate coordinate system is required to expose the structure, and this is highly non-trivial. Similar properties of classical solutions are also found in more general setups, such as in three dimension and in (A)dS spacetime. It would be very interesting to see how general this classical double copy structure can be.

Verifying the color-kinematics duality at higher loops is also very important. Although there are many progress in this direction, and we have the Kerr-Schild double copy that works as an non-perturbative example, it still remains a challenge to have an all-order proof. One possible way to do so is to expose the double copy structure already at the level of Lagrangian, using techniques such as double field theory. However, because of the gauge dependence and the off-shell nature, it is tricky to find a way to express a Lagrangian that features factorization of the Lorentz indices. One might need to construct it order-by-order, and even need to introduce auxiliary field to localize the interaction terms.

It is also essential to understand the algebra that governs the kinematic BCJ numerators. This is firstly understood for the self-dual sector of YM theory, where the algebra is identified with the area-preserving diffeomorphisms [159]. Recent progress has found a way to generate all tree-level BCJ numerators, either from a heavy-mass effective theory [160] or from the equations of motion for field strengths and currents [161]. The kinematic numerators is related to a quasi-shuffle Hopf algebra. Further research of the properties of this algebra is desirable.

It is still promising to apply QFT methods to computation in gravitational-wave physics. Gravitational scattering amplitudes at tree-level are obtained via double copy to construct higher loop contribution to PM amplitudes [45, 162]. However, it is desirable to attain loop amplitudes directly from the double copy. This requires to remove the additional dilaton field, which is not yet clear.

Appendix A

Circular orbits for the spinning YM potential

The goal of this section is to recap some basic notions of some linear algebra that are useful to understand the nature of the roots of a third and fourth order degree univariate polynomial, which are useful to solve the geodesic equations. For the polynomial

$$\lambda_1 x^4 + \lambda_2 x^3 + \lambda_3 x^2 + \lambda_4 x + \lambda_5 = 0, \tag{A.1}$$

the explicit roots are given by

$$\begin{aligned} x_{1,2} &= -\frac{\lambda_2}{4\lambda_1} - S \pm \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}} \\ x_{3,4} &= -\frac{\lambda_2}{4\lambda_1} + S \pm \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}, \end{aligned} \tag{A.2}$$

where we have defined

$$\begin{aligned} p &:= \frac{8\lambda_1\lambda_3 - 3\lambda_2^2}{8\lambda_1^2} \\ q &:= \frac{\lambda_2^3 - 4\lambda_1\lambda_2\lambda_3 + 8\lambda_1^2\lambda_4}{8\lambda_1^3} \\ S &:= \frac{1}{2}\sqrt{-\frac{2}{3}p + \frac{1}{3\lambda_1}\left(Q + \frac{\Delta_0}{Q}\right)} \\ Q &:= \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}} \\ \Delta_0 &:= \lambda_3^2 - 3\lambda_2\lambda_4 + 12\lambda_1\lambda_5 \\ \Delta_1 &:= 2\lambda_3^3 - 9\lambda_2\lambda_3\lambda_4 + 27\lambda_2^2\lambda_5 + 27\lambda_1\lambda_4^2 - 72\lambda_1\lambda_3\lambda_5. \end{aligned} \tag{A.3}$$

It turns out that if the discriminant [163]

$$\Delta := \frac{1}{27}\left(4\Delta_0^3 - \Delta_1^2\right) < 0, \tag{A.4}$$

then the equation (A.1) has two distinct real roots and two complex (conjugate) roots. In the particular case $\lambda_1 = 0$, we can use the reduced discriminant Δ_R

$$\Delta_R := \lambda_3^2\lambda_4^2 - 4\lambda_2\lambda_4^3 - 4\lambda_3^3\lambda_5 + 18\lambda_2\lambda_3\lambda_4\lambda_5 - 27\lambda_2^2\lambda_5^2 \tag{A.5}$$

which is positive when the third order degree polynomial has three real roots and negative when it has one real and two complex conjugate roots. We can apply these tools to find how many real solutions we have for the x variable in the case of circular orbits in $\sqrt{\text{Kerr}}$. Using from the polynomial equation (4.56) where

$$\begin{aligned}
\lambda_1 &:= mu \\
\lambda_2 &:= 4aL_{\text{crit}}mu^2\left(a^2u^2+1\right) \\
\lambda_3 &:= -L_{\text{crit}}^2 mu\left(a^2u^2+1\right)-2a^2mu \\
\lambda_4 &:= 4aL_{\text{crit}}m\left(a^2u^2+1\right) \\
\lambda_5 &:= a^4mu-\frac{L_{\text{crit}}^2 m\left(a^2u^2+1\right)}{u}.
\end{aligned} \tag{A.6}$$

we find that

$$\begin{aligned}
\Delta = &-16m^6\left(a^2u^2+1\right)^4\left(2a^2L_{\text{crit}}u^2\left(2a^2+L_{\text{crit}}^2\right)+L_{\text{crit}}^3\right)^2 \\
&\times\left\{L_{\text{crit}}^2u^2\left[27a^4u^4+L_{\text{crit}}^2a^2u^4+36a^2u^2+L_{\text{crit}}^2u^2+8\right]+16\right\}
\end{aligned} \tag{A.7}$$

is always manifestly negative.

Appendix B

Derivatives of the 3δ Integral in PM expansion

In this appendix we explicitly evaluate the expressions for the second order derivatives of the triple-delta integral $I_{3\delta}$ for $\sigma^2 > 0$, cf. (5.35). These enter into the three-body effective potential via (5.12). A priori we find four terms

$$\begin{aligned}\partial_1^\mu \partial_2^\nu I_{3\delta}(\sigma^2 > 0) = &\left(\partial_1^\mu \partial_2^\nu \frac{\pi}{4\sigma}\right) \Theta\left(-R_1^2 R_2^2 R_3^2\right) + \left(\partial_1^\mu \frac{\pi}{4\sigma}\right) \partial_2^\nu \Theta\left(-R_1^2 R_2^2 R_3^2\right) \\ &+ \left(\partial_2^\nu \frac{\pi}{4\sigma}\right) \partial_1^\mu \Theta\left(-R_1^2 R_2^2 R_3^2\right) + \frac{\pi}{4\sigma} \partial_1^\mu \partial_2^\nu \Theta\left(-R_1^2 R_2^2 R_3^2\right), \end{aligned} \tag{B.1}$$

which evaluate to

$$\begin{aligned}\Theta\left(-R_1^2 R_2^2 R_3^2\right) \partial_1^\mu \partial_2^\nu \frac{\pi}{4\sigma} = &\frac{\pi\Theta\left(-R_1^2 R_2^2 R_3^2\right)}{4\sigma^5}\Big[3\left(R_1 \cdot R_2\right)\big(R_1^\mu R_1^\nu R_2^2 + R_2^\mu R_2^\nu R_1^2 \\ &- R_1^\mu R_2^\nu R_1 \cdot R_2 - R_2^\mu R_1^\nu R_1 \cdot R_2\big) + \sigma^2\left(\eta^{\mu\nu} R_1 \cdot R_2 + R_1^\mu R_2^\nu + R_2^\mu R_1^\nu\right)\Big],\end{aligned} \tag{B.2}$$

$$\begin{aligned}\frac{\pi}{4\sigma} \partial_1^\mu \partial_2^\nu \Theta\left(-R_1^2 R_2^2 R_3^2\right) = &\frac{\pi}{4\sigma}\left[2\eta^{\mu\nu} \delta(R_3^2) + 4R_3^\mu R_3^\nu \delta'(R_3^2)\right] \mathrm{sgn}(R_1^2 R_2^2) \\ &+ \frac{8\pi}{4\sigma}\Big[R_2^\mu R_1^\nu \delta(R_1^2)\delta(R_2^2)\mathrm{sgn}(R_3^2) - R_2^\mu R_3^\nu \delta(R_2^2)\delta(R_3^2)\mathrm{sgn}(R_1^2) \\ &- R_3^\mu R_1^\nu \delta(R_1^2)\delta(R_3^2)\mathrm{sgn}(R_2^2)\Big],\end{aligned} \tag{B.3}$$

$$\begin{aligned}\left(\partial_1^\mu \frac{\pi}{4\sigma}\right) \partial_2^\nu \Theta\left(-R_1^2 R_2^2 R_3^2\right) = &\frac{\pi}{4\sigma}\left[\frac{4R_1^\mu R_1^\nu}{R_2^2 - R_3^2} \delta(R_1^2)\mathrm{sgn}(R_2^2 R_3^2)\right. \\ &\left. - \frac{4R_2^\mu R_3^\nu}{R_1^2 - R_2^2}\delta(R_3^2)\mathrm{sgn}(R_1^2 R_2^2) + R_3^\mu R_3^\nu \frac{R_1^2 + R_2^2}{\sigma^2}\delta(R_3^2)\mathrm{sgn}(R_1^2 R_2^2)\right].\end{aligned} \tag{B.4}$$

Here the last line also enters into (B.1) with the labels 1 and 2 interchanged. Note the appearance of the derivative of the delta function in the first line of (B.3) that one could resolve using

$$\delta'(R_3^2) = -\frac{\delta(R_3^2)}{R_3^2}. \tag{B.5}$$

Putting these terms together, eqn. (B.1) then becomes (ordered by the number of delta functions)

$$\partial_1^\mu \partial_2^\nu I_{3\delta} = \frac{\pi\Theta(-R_1^2 R_2^2 R_3^2)}{4\sigma^5}\Big[3\left(R_1 \cdot R_2\right)\left(R_1^\mu R_1^\nu R_2^2 + R_2^\mu R_2^\nu R_1^2 - R_1^\mu R_2^\nu R_1 \cdot R_2 - R_2^\mu R_1^\nu R_1 \cdot R_2\right)$$

$$
\begin{aligned}
&+\sigma^2\left(\eta^{\mu\nu}R_1\!\cdot\!R_2 + R_1^\mu R_2^\nu + R_2^\mu R_1^\nu\right)\Big] + \frac{\pi}{4\sigma}\left[\frac{4R_1^\mu R_1^\nu}{R_2^2 - R_3^2}\delta(R_1^2)\mathrm{sgn}(R_2^2 R_3^2)\right.\\
&\left.+\frac{4R_2^\mu R_2^\nu}{R_1^2 - R_3^2}\delta(R_2^2)\mathrm{sgn}(R_1^2 R_3^2)\right] + \frac{\pi}{4\sigma}\left[-\frac{4R_1^\mu R_3^\nu}{R_2^2 - R_1^2}\delta(R_3^2) - \frac{4R_2^\mu R_3^\nu}{R_1^2 - R_2^2}\delta(R_3^2)\right.\\
&\left.+2R_3^\mu R_3^\nu\frac{R_1^2 + R_2^2}{\sigma^2}\delta(R_3^2) + 2\eta^{\mu\nu}\delta(R_3^2) + 4R_3^\mu R_3^\nu\delta'(R_3^2)\right]\mathrm{sgn}(R_1^2 R_2^2)\\
&+\frac{8\pi}{4\sigma}\Big[R_2^\mu R_1^\nu\delta(R_1^2)\delta(R_2^2)\mathrm{sgn}(R_3^2) - R_2^\mu R_3^\nu\delta(R_2^2)\delta(R_3^2)\mathrm{sgn}(R_1^2)\\
&-R_3^\mu R_1^\nu\delta(R_1^2)\delta(R_3^2)\mathrm{sgn}(R_2^2)\Big]. \qquad \text{(B.6)}
\end{aligned}
$$

Performing the PN expansion starting from this expression seems (also conceptually) much harder than working on the level of the integrand of $I_{3\delta}$ in (5.12). The latter is demonstrated in subsection 5.4.2.

Appendix C

Details on 3PN of the three-body potential

In the computation of the 3PN potential, we added the following total derivative to remove the dependence on the derivative of accelerations and possible spurious poles for $r_{ij} \to \infty$:

$$L^{\rm td} = \sum_{j\neq i} \frac{G m_i m_j}{48\, c^8} \frac{\mathrm{d}}{\mathrm{d}t} \Big[r_{ij} (21\, \mathbf{a}_i \mathbf{v}_j - 18\, \mathbf{a}_i \mathbf{v}_i) \left((\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 + \mathbf{v}_j^2 \right) + r_{ij}\, \mathbf{n}_{ij} \mathbf{a}_i \mathbf{n}_{ij} \mathbf{v}_j \left((\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 - 3 \mathbf{v}_j^2 \right) \Big]. \tag{C.1}$$

Due to its length, here we display only an excerpt of the genuine three-body contribution to the 3PN effective potential from the third line of (5.12). The full result is given in an ancillary file. The expression below is organized according to the rational functions of the spatial distances, where each function is multiplied by a sum of numerator structures that scale as v^4. Note that some numerator structures begin with the same terms but they do not agree. Evaluating the derivatives in (5.77) yields the expression

$$
\begin{aligned}
L^{\rm 3PN}_{(D)} = & \sum_{j\neq i} \sum_{k \neq i,j} G^2 m_i m_j m_k \times \\
\Bigg\{ & \frac{1}{(r_{ij}+r_{jk}+r_{ik})^2} \left((\mathbf{n}_{ik}\cdot\mathbf{v}_i) \left(\frac{16}{3} (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^3 - 12 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 (\mathbf{n}_{ij}\cdot\mathbf{v}_j) + \frac{20}{3} (\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 \right) + 245 \text{ terms} \right) \\
& + \frac{1}{r_{ij}(r_{ij}+r_{jk}+r_{ik})} \left(\mathbf{v}_i^2 \left(\frac{16}{3} (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 - 11 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + \frac{16}{3} (\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 \right) + 45 \text{ terms} \right) \\
& - \frac{r_{ij}}{(r_{ij}+r_{jk}+r_{ik})^3} \left(\frac{8}{3} (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^4 - 6 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^3 (\mathbf{n}_{ij}\cdot\mathbf{v}_j) + \frac{8}{3} (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 (\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 + 286 \text{ terms} \right) \\
& - \frac{r_{ik} r_{jk}}{r_{ij}(r_{ij}+r_{jk}+r_{ik})^3} \left((\mathbf{n}_{ik}\cdot\mathbf{v}_k)^2 \left(16 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 - 36 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + 16 (\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 \right) + 69 \text{ terms} \right) \\
& - \frac{r_{ij}^2}{(r_{ij}+r_{jk}+r_{ik})^4} \left(8 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^4 - 18 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^3 (\mathbf{n}_{ij}\cdot\mathbf{v}_j) + 16 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 (\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 + 143 \text{ terms} \right) \\
& + \frac{r_{ik} r_{jk}}{(r_{ij}+r_{jk}+r_{ik})^4} \left((\mathbf{n}_{ik}\cdot\mathbf{v}_k)^2 \left(16 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 - 36 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + 16 (\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 \right) + 114 \text{ terms} \right) \\
& - \left[\frac{1}{r_{ik}(r_{ij}+r_{jk}+r_{ik})} \left(\frac{4}{3} \mathbf{v}_i^4 - \frac{4}{3} \left((\mathbf{n}_{ik}\cdot\mathbf{v}_i)^2 - (\mathbf{n}_{ik}\cdot\mathbf{v}_k)^2 \right) \mathbf{v}_i^2 + 20 \text{ terms} \right) + (i \leftrightarrow j) \right] \\
& - \left[\frac{r_{ik}}{r_{jk}(r_{ij}+r_{jk}+r_{ik})^2} \left((\mathbf{n}_{jk}\cdot\mathbf{v}_k)^2 \left(\frac{8}{3} (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 - 6 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + \frac{8}{3} (\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 \right) + 23 \text{ terms} \right) + (i \leftrightarrow j) \right] \\
& + \left[\frac{r_{ij}}{r_{ik}(r_{ij}+r_{jk}+r_{ik})^2} \left((\mathbf{n}_{jk}\cdot\mathbf{v}_k)^2 \left(\frac{8}{3} (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 - 6 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + \frac{8}{3} (\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 \right) + 47 \text{ terms} \right) + (i \leftrightarrow j) \right] \\
& - \left[\frac{r_{ik}}{r_{ij}(r_{ij}+r_{jk}+r_{ik})^2} \left(8 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^4 - 18 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^3 (\mathbf{n}_{ij}\cdot\mathbf{v}_j) + 12 (\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 (\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 + 93 \text{ terms} \right) + (i \leftrightarrow j) \right]
\end{aligned}
$$

$$
\begin{aligned}
&- \left[\frac{r_{ik}}{(r_{ij}+r_{jk}+r_{ik})^3}\left(16(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^4 - 36(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^3(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + 24(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 + 285 \text{ terms}\right) + (i \leftrightarrow j)\right] \\
&+ \left[\frac{r_{ik}^2}{r_{jk}(r_{ij}+r_{jk}+r_{ik})^3}\left((\mathbf{n}_{jk}\cdot\mathbf{v}_k)^2\left(\frac{16}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 - 12(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + \frac{16}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\right) + 46 \text{ terms}\right) + (i \leftrightarrow j)\right] \\
&- \left[\frac{r_{ij}^2}{r_{ik}(r_{ij}+r_{jk}+r_{ik})^3}\left((\mathbf{n}_{ik}\cdot\mathbf{v}_i)^2\left(\frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 - 6(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + \frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\right) + 58 \text{ terms}\right) + (i \leftrightarrow j)\right] \\
&- \left[\frac{r_{ij}r_{jk}}{r_{ik}(r_{ij}+r_{jk}+r_{ik})^3}\left((\mathbf{n}_{ik}\cdot\mathbf{v}_i)^2\left(\frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 - 6(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + \frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\right) + 72 \text{ terms}\right) + (i \leftrightarrow j)\right] \\
&- \left[\frac{r_{ik}^2}{r_{ij}(r_{ij}+r_{jk}+r_{ik})^3}\left((\mathbf{n}_{ik}\cdot\mathbf{v}_k)^2\left(\frac{40}{3}(\mathbf{n}_{ik}\cdot\mathbf{v}_i)^2 - 30(\mathbf{n}_{ik}\cdot\mathbf{v}_i)(\mathbf{n}_{ik}\cdot\mathbf{v}_j) + \frac{40}{3}(\mathbf{n}_{ik}\cdot\mathbf{v}_j)^2\right) + 75 \text{ terms}\right) + (i \leftrightarrow j)\right] \\
&+ \left[\frac{r_{ik}^2}{(r_{ij}+r_{jk}+r_{ik})^4}\left((\mathbf{n}_{ik}\cdot\mathbf{v}_k)^2\left(16(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 - 36(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + 16(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\right) + 109 \text{ terms}\right) + (i \leftrightarrow j)\right] \\
&- \left[\frac{r_{ij}r_{ik}}{(r_{ij}+r_{jk}+r_{ik})^4}\left(16(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^4 - 36(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^3(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + 24(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2 + 174 \text{ terms}\right) + (i \leftrightarrow j)\right] \\
&- \left[\frac{r_{ik}^2(r_{ij}+r_{jk})}{r_{ij}r_{jk}(r_{ij}+r_{jk}+r_{ik})^3}\left((\mathbf{n}_{jk}\cdot\mathbf{v}_k)^2\left(\frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 - 6(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + \frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\right) + 21 \text{ terms}\right) + (i \leftrightarrow j)\right] \\
&+ \left[\frac{r_{ik}^3(r_{ik}+2r_{jk})}{r_{ij}r_{jk}(r_{ij}+r_{jk}+r_{ik})^4}\left((\mathbf{n}_{ik}\cdot\mathbf{v}_k)^2\left(8(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2 - 36(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j) + 8(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\right) + 33 \text{ terms}\right) + (i \leftrightarrow j)\right]\Bigg\}.
\end{aligned}
\tag{C.2}
$$

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